Analysis

with ultrasmall numbers

Standard Level

PART I

This version has proofs and comments for the teacher

These come in frames like this one, and for this reason, the page numbers are not the same as on the student handout version.

The proofs given must not be understood as "the" proofs, but as the ones which over the years, I feel most comfortable with. When a theorem does not need anything specific to ultracalculus, the proof is omitted.

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Infinity itself looks flat and uninteresting. [...] The chamber [...] was anything but infinite, it was just very very very big, so big that it gave the impression of infinity far better than infinity itself.
(Douglas Adams: The Hitchhiker's Guide to the Galaxy)

Contents

1 Introduction

1.1 Velocity and Position

Exercise 1

Suppose the velocity [1] of a car is constant and equal to $60km/h$.

(1) Let f be the function which describes the position of the car with respect to time. Draw the graph f for t ranging from 0 to 3 hours.

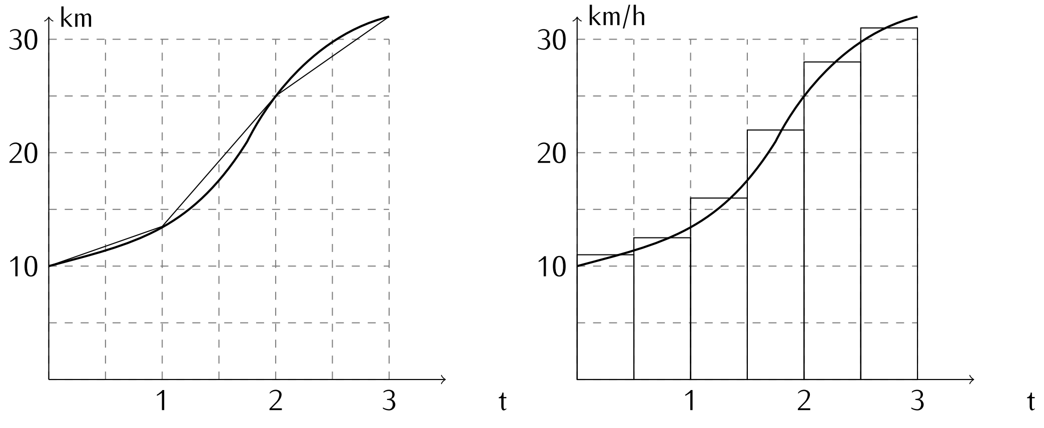
(2) Let v be the function which describes the velocity of the car with respect to time. Draw the graph of v for t ranging from 0 to 3 hours.

(3) Given the position graph, how can one find the velocity of the car at any given time?

(4) Given the velocity graph, how can one find the position of the car after any given time?

⚠ Note the difference: velocity (deduced from position) is *local*. It is possible to give the velocity *at* a given time. Position (deduced from velocity) is *global*. It is only possible to find the *variation* of the position over an *interval* of time.

A curve can be approximated by a piecewise linear function whose slope is easily calculated by pieces. It can also approximated by a "staircase" function whose area is calculated by adding the areas of the rectangles.



[1] The velocity is speed with a direction. Speed is always positive (or zero); velocity can be negative.

The main goal of the subject called **mathematical analysis** will be to check when and how to approximate a curve by pieces of straight lines and when and how to approximate areas by rectangles and to understand what these can be used to calculate. Intuitively, it should seem clear that in order for the approximation to be good, the pieces of straight lines or the rectangles must be small – or that the number of pieces is large. The crucial questions are: How small? and How large?

1.2 Tiny and Huge

Exercise 2

If δ is a positive value which is really tiny (even tinier than that!),

(1) what can you say about the size of δ^2, $2 \cdot \delta$ and $-\delta$?

(2) what can you say about $2 + \delta$ and $2 - \delta$?

(3) what can you say about $\frac{1}{\delta}$?

> Note for the teacher: there is no “tending to”. δ *is* tiny; just as its reciprocal is huge. Note that “tiny” must be defined to be small in absolute value, since -10^{10} is smaller that 5...
> *“tending to” yields an informal metaphor of x moving towards a: recall that numbers do not move...*

Exercise 3

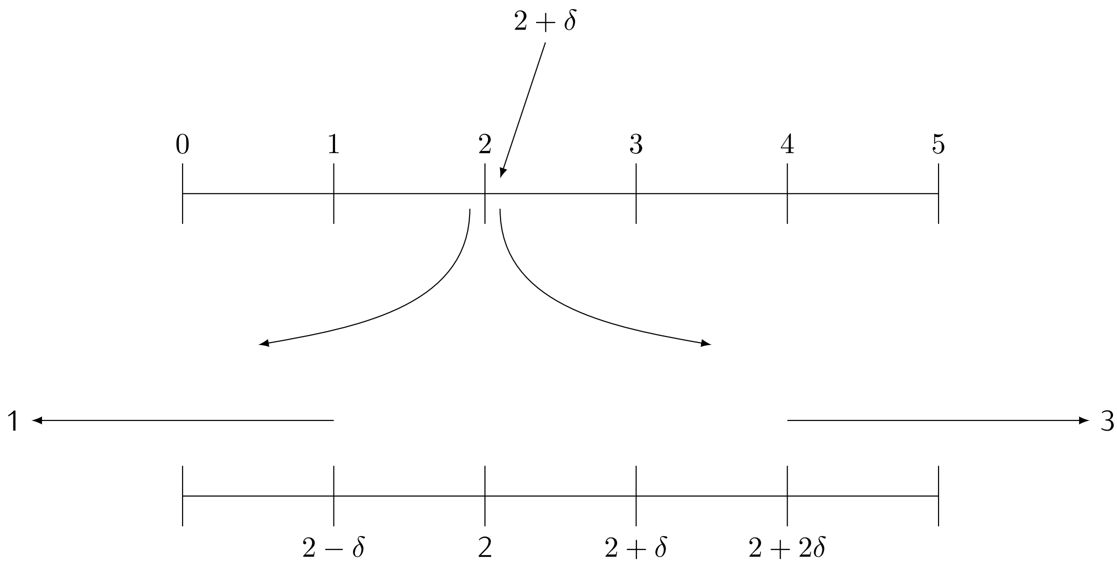
If N is a positive huge number (really very huge!),

(1) what can you say about N^2, $2N$ and $-N$?

(2) what can you say about $N + 2$ and $N - 2$?

(3) what can you say about $\frac{1}{N}$?

(4) what can you say about $\frac{N}{2}$?



Zooming in

Exercise 4

Let $f : x \mapsto x^2$, and let δ be vanishingly small and positive.

(1) Draw the result of a zoom centred on $\langle 1; 1\rangle$ of f so that δ becomes visible.

Show, on the drawing, the values 1 and $f(1)$, $1 + \delta$ and $f(1 + \delta)$, $1 - \delta$ and $f(1 - \delta)$.

What does the curve look like?

(2) For the same function, draw the result of a zoom centred on $\langle 2; 4\rangle$

Show, on the drawing, the values 2 and $f(2)$, $2 + \delta$ and $f(2 + \delta)$, $2 - \delta$ and $f(2 - \delta)$.

(3) Similar question for a zoom centred on $\langle 0; 0\rangle$.

(4) Similar question for a zoom centred on $\langle -1; 1\rangle$.

Exercise 5

Draw the result of zooms so that a tiny δ becomes visible for
$g : x \mapsto x^3$, and $h : x \mapsto |x|$
For g: centres are $\langle 1; 1\rangle$, $\langle -2; -8\rangle$ and $\langle 0; 0\rangle$
For h: centres are $\langle 1; 1\rangle$, $\langle -2; 2\rangle$ and $\langle 0; 0\rangle$

Exercise 6

Draw a zoom centred on $\langle 0;0\rangle$ and another zoom centred on $\langle 0;-1\rangle$ for

$$k : x \mapsto \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

When we say that δ is "tiny", we want it to be tiny compared to all the parameters involved; this leads to the following definition:

Definition 1
*The **context** of a property, function or set is the list of parameters used in its definition. The context can be a single number.*

A context is *extended* if parameters are added to the list.

Before defining more precisely what it means to be "tiny" we must first clarify what it means to be observable:

Observability

(1) Numbers defined without reference to observability are always observable – or standard.

(2) If a is not observable in the context of b, then b is be observable in the context of a. (the context from which both are observable is the common context).

(3) **Closure:** If a number satisfies a given property, then there is an observable number satisfying that property.

(4) A property referring to observability is true if and only if it is true when its context is extended.

A consequence of (3) is that the results of operations between two numbers are in their common context.

The word "observable" , by convention, refers to a context. Informally: the context is the parameters, sets and functions the statement is about. Therefore to determine the context of a statement, one must be able to understand it and describe what it says and about what it says something.

But: a consequence of (4) is that it does not matter what the context is precisely provided it contains at least all parameters involved.

All "familiar" numbers such as 1; 3; 10^{10}; $\sqrt{2}$ or π are always observable, or standard, but also – in general –

$$\boxed{f(a) \quad \text{is observable}}$$

This refers to the context, by the word "observable". The only parameters of this property are f and a. This is the context.

Non observable values do not show up unless explicitly summoned.

Definition 2
*A real number is **ultrasmall** if it is nonzero and smaller in absolute value than any strictly positive observable number*

This definition makes an implicit reference to a context.

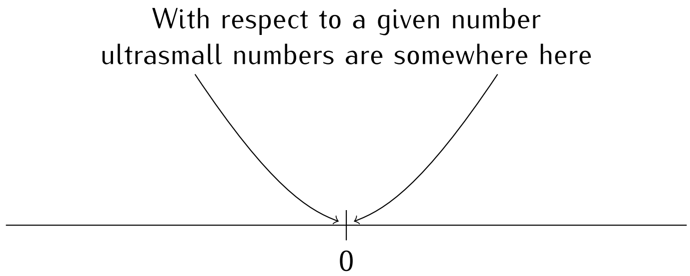
⚠ Note that 0 is not ultrasmall.

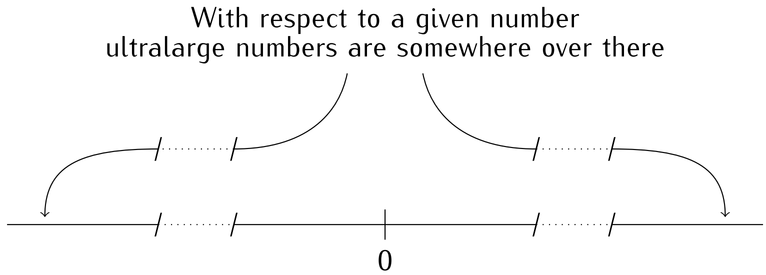
> **Principle of ultrasmallness**
> *Relative to any context, there exist ultrasmall real numbers.*

Such an ultrasmall number is then part of an **extended context**.
Given a context, if ε is ultrasmall then ε is not observable.

Definition 3
*A real number is **ultralarge** if it is larger in absolute value than any strictly positive observable number*

⚠ Note the asymmetry: if h is ultrasmall relative to x, then it is not observable. But x is observable relative to h (see the third item of the observability pricniple), hence x **is not** ultralarge relative to h.





Definition 4
*Let a, b be real numbers. We say that a is **ultraclose** to b, written*

$$a \simeq b,$$

if $b - a$ is ultrasmall or if $a = b$.

This definition makes an implicit reference to a context.
In particular, $x \simeq 0$ if x is ultrasmall or zero.

If $a \simeq b$ then a and b are said to be neighbours. If a is a neighbour of b and is observable (relative to some context) then a is the observable neighbour of b.

Theorem 1
Relative to a context: If a and b are observable and $a \simeq b$, the $a = b$.

Exercise 7
Prove the previous theorem. (you will need to refer to closure)

If $a \simeq b$ then $a - b \simeq 0$; which means that $a - b$ is ultrasmall or zero. By closure, it is observable, hence cannot be ultrasmall.

A rational number may have an observable neighbour which is not rational. The number $\sqrt{2}$ is always observable because it is completely and uniquely defined by the parameter 2. Relative to this context consider an ultralarge N and take the first N digits of $\sqrt{2}$. This is a rational number which is not observable. Yet it is ultraclose to an observable number which is $\sqrt{2}$.

The existence of an observable neighbour is given by the following

Principle of the observable neighbour
Relative to a context, any real number x which is not ultralarge can be written in the form $a + h$ where a is observable and $h \simeq 0$.

Exercise 8
Show that if x has an observable part, then it is unique.

Assume a and b are observable neighbours, then $a \simeq x \simeq b \Rightarrow a \simeq b$ and by theorem 1, $a = b$.

This unique number is **the observable neighbour** of x.

Exercise 9
Prove the following:

Theorem 2
Let $[a; b]$ be an interval. Show that if x is in $[a; b]$, then the observable part of x is not outside $[a; b]$.

Assume by contradiction that $x \in [a, b]$ and that $c \simeq x$ is outside, and larger than b. We then have $x \leq b \leq c$ with $x \simeq c$. But this implies $b \simeq c$ so $b = c$. (Same for $c \leq a$.) *The observability is given by a and b.*

Exercise 10

Prove the following:

(1) If ε is ultrasmall relative to x then $\frac{1}{\varepsilon}$ is ultralarge relative to x.

(2) If M is ultralarge relative to x then $\frac{1}{M}$ is ultrasmall relative to x.

Exercise 11

Prove the following theorems (together they give all the rules needed for analysis and will be referred to by "ultracomputation" or "ultracalculus"):

Theorem 3
Let ε and δ be ultrasmall relative to a context and let a be observable and not zero.

(1) Then: $a \cdot \varepsilon$ is ultrasmall.

> By contradiction. Assume $a \cdot \varepsilon \not\simeq 0$. Then by definition, there is an observable strictly positive b such that $|a \cdot \varepsilon| = |a| \cdot |\varepsilon| \geq b > 0$. But then $|\varepsilon| \geq \frac{b}{|a|} > 0$. By closure $\frac{b}{|a|}$ is observable. This contradicts that ε is ultrasmall.
> *The proof by contradiction assumes the existence of (one) counterexample. A direct proof requires to show something about all observable positive numbers.*

(2) Then: $\varepsilon + \delta \simeq 0$

> $0 \leq |\varepsilon + \delta| \leq 2 \cdot \max\{|\varepsilon|, |\delta|\}$ which is two times an ultrasmall, whic is ultrasmall by the previous point.

(3) Then: $\varepsilon \cdot \delta$ is ultrasmall

> Obvious, but if necessary: $0 < |\delta| < 1$ so $0 < |\varepsilon \cdot \delta| < |\varepsilon|$

(4) If $a \neq 0$ Then: $\dfrac{a}{\varepsilon}$ is ultralarge

> Again by contradiction: assume it is not ultralarge, then there is an observable $b > 0$ such that $|\frac{a}{\varepsilon}| = \frac{|a|}{|\varepsilon|} < b \Rightarrow |a| < |b| \cdot |\varepsilon| \simeq 0$, which contradicts that a is observable.

> *The following properties can be proven later, when after some specific exercises, a general formula is need. Could be postponed to beginning of chapter 5.*

Theorem 4 (Ultracomputation)
Relative to a context: If a and b are observable and not zero and if $a \simeq x$ and $b \simeq y$,

(1) $a + b \simeq x + y$

(2) $a - b \simeq x - y$

> Write $x = a + \varepsilon, y = b + \delta$. Then $x+y = a+\varepsilon+b+\delta$ and since $\varepsilon+\delta \simeq 0$ by theorem 3 we have the conclusion.

(3) $a \cdot b \simeq x \cdot y$

> as before, then $x \cdot y = (a+\varepsilon)\cdot(b+\delta) = a \cdot b + a \cdot \delta + b \cdot \varepsilon + \varepsilon \cdot \delta \simeq a \cdot b$ by theorem 3

(4) If also $b \neq 0$, $\dfrac{a}{b} \simeq \dfrac{x}{y}$.

For the last item of theorem 4, it is enough to show:

Relative to a context. If b is observable and $b \neq 0$ and if $b \simeq y$ then $\dfrac{1}{b} \simeq \dfrac{1}{y}$

and use item 3 to conclude.

> Writing $y = b + \delta$ and $\frac{1}{y} = \frac{1}{b+\delta} = \frac{1}{b} + h$ leads to $b = (1 + bh)(\underbrace{b + \delta}_{\simeq b}) \simeq (1 + bh) \cdot b$, hence $1 + bh$ must be ultraclose to 1, so $bh \simeq 0$ and $h \simeq 0$.

> More tricky but more powerful: good for maths 2:
> b is observable and not zero, hence for $y \simeq b$, y is not ultrasmall nor ultralarge. Therefore $\frac{1}{y}$ is not ultralarge nor ultrasmall, hence it has an observable neighbour $c \simeq \frac{1}{y}$. We have $cy \simeq 1$ and then $\frac{1}{c} \simeq y \simeq b$. But by closure, $\frac{1}{c}$ is observable, so $\frac{1}{c} = b$. So $\frac{1}{y} \simeq \frac{1}{b}$.

Practice exercise 1 Answer page 15

Consider a context.

(1) Give an example of x and y such that $x \simeq y$ but $x^2 \not\simeq y^2$.

(2) Give an example of x and y such that $x \simeq y$ but $\frac{1}{x} \not\simeq \frac{1}{y}$.

Practice exercise 2 Answer page 15

Relative to a context.

In the following, assume that ε, δ are positive ultrasmall and H, K positive ultralarge numbers. Determine whether the given expression yields an ultrasmall number, an ultralarge number or a number in between.

(1) $1 + \dfrac{1}{\varepsilon}$

(2) $\dfrac{\sqrt{\delta}}{\delta}$

(3) $\sqrt{H+1} - \sqrt{H-1}$

(4) $\dfrac{H+K}{H \cdot K}$

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7}$

(6) $\dfrac{\sqrt{1+\varepsilon} - 2}{\sqrt{1+\delta}}$

Practice exercise 3 Answer page 15

Relative to a context find ultrasmall ε and δ (or the relation between them) such that $\dfrac{\varepsilon}{\delta}$ is:

(1) not ultralarge and not ultrasmall,

(2) ultralarge,

(3) ultrasmall.

⚠ The previous exercise show that if no relation is known between ultrasmall numbers ε and δ, their quotient can be of any possible magnitude.

Contextual Notation

The only acceptable properties are those that do not refer to observability or those that use the symbol "$\simeq$".

This is an extremely important restriction, even though it is probably not necessary to mention it otherwise than saying it is a rule which must be followed. The thing is that with ultrasmall numbers not any property can be used to determine a set. As a direct example: relative to the standard level, it is not possible to collect all ultrasmall numbers inot a set. If we could, we would have a set which is bounded above (by 1) but which has no least upper bound, which would contradict that all sets of real numbers bounded above have a least upper bound.
Recall that the context is the parameters that the statement is about. When we define a set by a property, this must state a property for the element to belong to the set, hence if ultrasmall values are invoqued they must be relative to the context containing the element, and it cannot be ultrasmall relative to itself.
In fact, ultrasmall values can nonly be used to determine a property such as in the derivative: they appear as "dummy variables".
Here is another example of what would go wrong:
Let $\mathbf{obs}_1(x)$ stand for the observable neighbour of x relative to the standard level. Consider the rule $x \mapsto \mathbf{obs}_1(x)$. If this defined a function, then zooming on the graph we would see a horizontal line on any ultrasmall neighbourhood (all points on an ultrasmall interval have the same observable neighbour.) There is no value where we could point to a discontinuity yet this everywhere horizontal "continuous" graph (if it exists) is increasing!
The problem here is the reference to a level not referring to the context containing x.

The Problem of Induction

For students, induction is not the natural way to think about mathematical objects (not yet). Some mathematicians are troubled by some nonstandard statements which seem to contradict induction. The question is addressed here.

Statements about observability are always relative to the context of the statement. (Contextual statements)

- Statements that do not refer to observability can be used in induction proofs (these are the classical induction proofs).
- Statements that use "$\simeq$" can also be used in induction proofs.
- Statements that use "standard" cannot be used in induction proofs since there is an absolute reference to a level independently of the context.

Thus even though it is true that if n is observable then $n+1$ is observable, one cannot deduce that all numbers are observable. This statement is about n, hence the context contains n. By the convention that observable always refers to the context, n is observable can be rewritten as n is as observable as itself – which is true! So by induction, we would get, at best, that every number is as observable as itself.

Answers to practice exercises

Answers to practice exercice 1, page 13

(1) Let $x = N$ be ultralarge, and $y = N + \frac{1}{N}$ so $x \simeq y$ but $x^2 = N^2 \not\simeq N^2 + 2 + \frac{1}{N^2} = y^2$.

(2) Let h be ultrasmall, then let $x = h$ and $y = h^2$. Then $x \simeq 0$ and $y \simeq 0$ hence $x \simeq y$. Then $\frac{1}{h}$ and $\frac{1}{h^2}$ are both ultralarge and $\frac{1}{h^2} - \frac{1}{h} = \frac{1}{h}(\frac{1}{h} - 1)$. By ultracomputation, this is ultralarge, hence $\frac{1}{x} \not\simeq \frac{1}{y}$.

Answers to practice exercice 2, page 13

The terms ultrasmall or ultralarge all refer to a given context.

(1) As $\frac{1}{\varepsilon}$ is ultralarge $1 + \frac{1}{\varepsilon}$ is ultralarge.

(2) We have $\frac{\sqrt{\delta}}{\delta} = \frac{1}{\sqrt{\delta}}$ which is ultralarge.

(If $\delta < c$ for any observable c, then $\sqrt{\delta} < \sqrt{c}$ and $\sqrt{\delta} \simeq 0$ hence $\frac{1}{\sqrt{\delta}}$ is ultralarge.)

(3) Maybe surprisingly, this is ultrasmall. To see this we multiply and divide by the conjugate:

$$\begin{aligned}\sqrt{H+1} - \sqrt{H-1} &= \frac{(\sqrt{H+1} - \sqrt{H-1})(\sqrt{H+1} + \sqrt{H-1})}{\sqrt{H+1} + \sqrt{H-1}} \\ &= \frac{(H+1) - (H-1)}{\sqrt{H+1} + \sqrt{H-1}} \\ &= \frac{2}{\sqrt{H+1} + \sqrt{H-1}}.\end{aligned}$$

H is assumed positive, its square root (plus or minus 1) is also a positive ultralarge. The sum of 2 positive ultralarge numbers is ultralarge hence the quotient is ultrasmall.

(4) $\dfrac{H+K}{HK} = \dfrac{1}{K} + \dfrac{1}{H}$ is ultrasmall.

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7} = \dfrac{35 + 7\varepsilon - 35 - 5\delta}{49 + 7\delta} = \dfrac{\overbrace{7\varepsilon - 5\delta}^{\simeq 0}}{\underbrace{49 + 7\delta}_{\simeq 49}}$ is ultrasmall or zero.

(6) $\dfrac{\overbrace{\sqrt{1+\varepsilon} - 2}^{\simeq -1}}{\underbrace{\sqrt{1+\delta}}_{\simeq 1}} \simeq -1$, hence not ultralarge and not ultrasmall.

Answers to practice exercice 3, page 13

(1) Take $\varepsilon = \delta$ then $\dfrac{\varepsilon}{\delta} = 1$.

(2) Take $\delta = \varepsilon^2$, then $\dfrac{\varepsilon}{\delta} = \dfrac{1}{\varepsilon}$ is ultralarge.

(3) Take $\varepsilon = \delta^2$, then $\dfrac{\varepsilon}{\delta} = \delta$ is ultrasmall.

2
Derivatives

> *The goal of this chapter is to give meaning to the definition of the derivative. Nothing about the derivative will be assumed other than the definition. Differentiation rules will be proved later.*
> *This chapter contains no theorem and no proof.*

We will often use Δx to indicate an ultrasmall *increment*[1] of the variable x. It may be positive or negative but will never be chosen to be 0.

Exercise 12
Let

$$f : x \mapsto x^2$$

The graph of this function is a curve (a parabola). Zoom in on the point $\langle 2, 4\rangle$. 2 and 4 are always observable. Consider the value of the function at $2+\Delta x$, (for Δx ultrasmall as mentioned above) and draw a straight line passing through $\langle 2, 4\rangle$ and $\langle 2+\Delta x, f(2+\Delta x)\rangle$.

- What is the slope of this straight line?
- What observable value is this slope ultraclose to?

Definition 5
A real function f defined on an interval containing a is ***differentiable at*** *a if there is an observable value D such that*

$$\frac{f(a+\Delta x) - f(a)}{\Delta x} \simeq D$$

not depending on ultrasmall Δx.
Then $D = f'(a)$ is the ***derivative*** *of f at a.*

When the derivative exists, it is the observable neighbour of $\dfrac{f(a+\Delta x) - f(a)}{\Delta x}$.

Metaphorically, finding the derivative can be described by: Zoom in. If what you see is indiscernible from a straight line, then measure the slope of that line. Zoom out. Drop what you can't see.

[1] increment: a positive or negative change in a variable.

Exercise 13

Using definition 5 calculate the derivative of: $f : x \mapsto 3x^2 + x - 5$ at $x = -2$ and $x = 2$.

Exercise 14

Using definition 5 calculate the derivatives (if they exist) of the following:

(1) $g : x \mapsto 2x^3 - 2$ at $x = 1$ and $x = 0$.

(2) $h : x \mapsto |x|$ at $x = 2$, $x = -2$ and at $x = 0$.

Exercise 15

Let $f : x \mapsto x^3 - 3x - 2$. Check that 2 is a root of f. Are there other roots?

At what values of x is the derivative equal to zero? What is the value of the function at these points? At what values of x de we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

Exercise 16

Let $f : x \mapsto 2x^3 - 4x^2 + 2x$. At what values of x is the function equal to zero? At what values of x is the derivative equal to zero? What is the value of the function at these points? At what values of x de we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

Practice exercise 4 Answer page 28

Calculate the derivative of the following:

(1) $f : x \mapsto 5x^2 - 10x$ at $x = 2$

(2) $g : x \mapsto 5(x - 10)^2$ at $x = 3$

(3) $h : x \mapsto x^4 + x^3 + x^2 + x + 1$ at $x = 1$

(4) $k : x \mapsto 5x^2 + 10$ at $x = 2$

Exercise 17

Consider the derivative at x (general case) of the function

$$f : x \mapsto x^2 + 3x.$$

Show that it is differentiable for all x and that $f'(x) = 2x + 3$.

Notice that in a derivative, if there is one, the division is **always** between two ultrasmall numbers. They <u>cannot</u> be replaced by 0 since $\frac{0}{0}$ is not defined.

3
Continuity

Informally: a function is continuous if it is where you would expect it to be by observing where it is just before and just after.

Definition 6 (Continuity)
Let f be a real function defined around a. We say that f ***is continuous at*** *a if*

$$x \simeq a \Rightarrow f(x) \simeq f(a).$$

The continuity of f at a is a property of f and a. Hence the context is given by f and a.

Exercise 18
Show that $f : x \mapsto x^3$ is continuous at $a = 2$.

Exercise 19
Show whether $f : x \mapsto \dfrac{x}{x^2+1}$ is continuous for all values of x.

Exercise 20

(1) Show that $f : x \mapsto |x|$ is continuous at $x = 0$, at $x = 1$, at $x = -1$ and at x in general.

(2) Show that $g :\mapsto \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$ is continuous at $x = 0$ and at x in general.

(3) Show that $g :\mapsto \begin{cases} x^2 & \text{if } x \geq -1 \\ x^3 & \text{if } x < -1 \end{cases}$ is not continuous at $x = -1$ but is continuous for all other values of x.

Exercise 21

Prove the following theorem:

Theorem 5

Let f and g be two real functions continuous at a. Then

(1) $f \pm g$ *is continuous at* a.

(2) $f \cdot g$ *is continuous at* a.

(3) $\dfrac{f}{g}$ *is continuous at* a *if* $g(a) \neq 0$.

For $x \simeq a$, we have $f(x) \simeq f(a)$ and $g(x) \simeq g(a)$. The conclusions follow by theorem 4.

It is also possible to introduce dependent variables u and v.
$f(a) = b$, $g(a) = c$, $f(x) = u$ and $g(x) = v$
By continuity $b \simeq u$ and $c \simeq v$
By theorem 4, $b \pm c \simeq u \pm v$ $\quad b \cdot c \simeq u \cdot v$ and $\frac{b}{c} \simeq \frac{u}{v}$.

Exercise 22

Prove the following theorem:

Theorem 6

Let f and g be two real functions. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

$g(x) \simeq g(a)$ hence $f(g(x)) \simeq f(g(a))$. And that is it.

So short that maybe some expanding may help (useful to prepare the way for the chain rule).
Let $g(a) = b$ and $g(x) = u$. By continuity of g at a, we have $b \simeq u$ and by continuity of f at b, we have $f(b) \simeq f(u)$.

A function f is defined on the left of a (resp. on the right) if $f(x)$ is defined for all $x \simeq a$ with $x < a$ (resp. $x > a$). It is clear that f is defined around a if and only if f is defined on the right and on the left of a.

We now extend the concept of continuity at a point to continuity on an interval.

Definition 7 (Continuity on an Interval)

(1) Let f be a real function defined on the open interval $]a;b[$. Then f is ***continuous on*** *$]a;b[$ if f is continuous at all $x \in]a;b[$.*

(2) Let f be a real function defined on the closed interval $[a;b]$. Then f is ***continuous on*** *$[a;b]$ if f is continuous at all $x \in]a;b[$ and if f continuous on the right at a and on the left at b.*

Informally: a function is continuous on an interval if its curve can be drawn without lifting the pencil, or if the function is where you expect it to be if it is hidden by a vertical line.

Exercise 23

Determine whether $f : x \mapsto x^2$ is continuous on its domain.

Clearly, if f and g are continuous on an interval I then the sum, difference, product and quotient (if $g(x) \neq 0$) are continuous on I. Moreover, if g is continuous on an interval containing $f(I)$ then $g \circ f$ is continuous on I.

Exercise 24

Show, using the definition of continuity, whether the following functions are continuous on the given intervals.

(1) $f_1 : x \mapsto \frac{1}{3}x + \sqrt{2}$ on $\mathbb{R}$

(2) $f_2 : x \mapsto x^2 - 3x - 1$ on $\mathbb{R}$

(3) $f_3 : x \mapsto \dfrac{x+2}{x-1}$ on $]1; +\infty[$

Exercise 25

Determine whether $f : x \mapsto \frac{1}{x}$ is continuous on its domain.

Exercise 26

Prove that $x \mapsto \sqrt{x}$ is continuous on its domain i.e, for any value $x = a$ in the domain.

4
Derivative Functions

Definition 8
If a function is differentiable on a given interval I, then for any $x \in I$ the value $f'(x)$ exists. Hence we can define ***the derivative function*** *by*

$$f' : x \mapsto f'(x)$$

If $f'(a) = 0$, then in an ultrasmall neighbourhood of a the function is **stationary** – on an ultrasmall neighbourhood $[a - \Delta x; a + \Delta x]$ its variation is of the form $\varepsilon \cdot \Delta x$ – its graph is indistinguishable from a horizontal line.

Exercise 27
Differentiate $f : x \mapsto x^2$ and $g : x \mapsto x^3$ at general x.

Notation: Let Δx be ultrasmall relative to f and x. We write

$$\Delta f(a) = f(a + \Delta x) - f(a) \text{ or } f(a + \Delta x) = f(a) + \Delta f(a).$$

Hence, we have:

$$\frac{\Delta f(a)}{\Delta x} \simeq f'(a).$$

Notation: A "$\simeq$" symbol may be replaced by a "$=$" symbol by adding a value ultraclose to zero on one of the sides i.e., $a \simeq b \Rightarrow a = b + \varepsilon$ where $\varepsilon \simeq 0$.

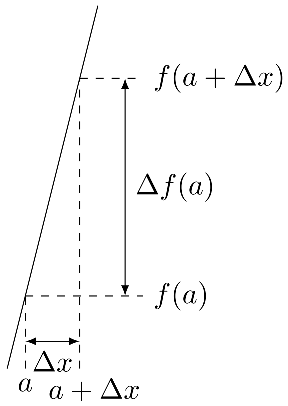
Hence

$$\frac{\Delta f(a)}{\Delta x} = f'(a) + \varepsilon \text{ with } \varepsilon \simeq 0$$

and we also have the form

$$\Delta f(a) = f'(a) \cdot \Delta x + \varepsilon \cdot \Delta x$$

which is called **the increment equation**.



Note: drawings involving ultrasmall or ultralarge values are not meant to be to scale nor be a correct representation. Their purpose is merely to help the mind.

Exercise 28
Prove the following theorem:

Theorem 7
If a real function f is differentiable at a then f is continuous at a.

(1) Give a direct proof.

> *We start using the power of the increment equation.*
> $\Delta f(a) = \underbrace{f'(a)\cdot\Delta x}_{\text{observable}\times\text{ultrasmall }\simeq 0} + \underbrace{\varepsilon\cdot\Delta x}_{\simeq 0}$

(2) Give a proof by contrapositive.

> Assume f is not continuous at a, then there is an $x \simeq a$ such that $f(x) \not\simeq f(a)$. So $|f(a)-f(x)| \geq b$, for some observable positive b. Then $\frac{|f(a)-f(x)|}{\delta x} \geq \frac{b}{\Delta x}$. This last term is ultralarge (observable/ultrasmall) so there is no observable neighbour, so no derivative.

Practice exercise 5 Answer page 28
Using definition 5, give the derivative functions of the following functions:

(1) $f : x \mapsto 3x+2$

(2) $g : x \mapsto 2x^2 - x$

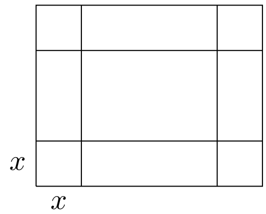
(3) $h : x \mapsto 5x^3 + 2x^2 - x$

(4) $k : x \mapsto 5x^3 + 2x^2 + 3x + 2$

In some cases, the slope to the right of a point is not the same as the slope to the left of that point. The derivative is the slope when it is the same on both sides.

Exercise 29

A factory wants to make cardboard boxes (with no top) out of sheets of $30cm \times 16cm$



The volume will be a function of x. The dimensions of the base are $30 - 2x$ and $16 - 2x$ (in centimetres). The height is x. What value(s) of x give(s) the maximum volume for the box?

4.1 Tangent line

Suppose f is differentiable at x_0. We observe that through a microscope, the curve of a function f at x_0 is indistinguishable from a straight segment. This straight segment meets the function at $\langle x_0; f(x_0)\rangle$ and there is a unique line which extends this segment with slope equal to the derivative which is indistinguishable from the curve. This line is the tangent line.

Definition 9

Let f be differentiable at x_0. The tangent line T_{x_0} is the unique line through $\langle x_0; f(x_0)\rangle$ with slope $f'(x_0)$.
It is the only straight line T which satisfies $T(x_0) = f(x_0)$ and $T'(x_0) = f'(x_0)$.

Exercise 30

Let $f : x \mapsto x^2$. Find the equation of the straight line tangent to f at $x = 3$.

Exercise 31

Show that

$$T_{x_0} : x \mapsto f'(x_0)(x - x_0) + f(x_0).$$

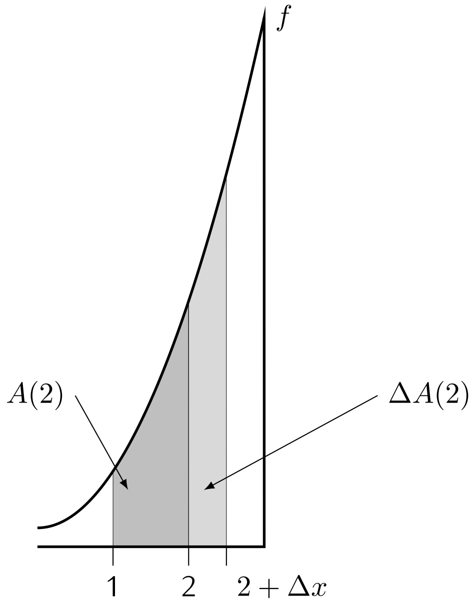
Exercise 32

Give the equation of the line tangent to $x \mapsto x^3 - 3 \cdot x^2$ at $x = 2$. For which values of x is this tangent horizontal?

4.2 Area under a curve

Consider a nonnegative function f continuous on a closed interval $[a; b]$. Note $A(x)$ the area between the curve of f and the horizontal x-axis.

The variation between x and $x + \Delta x$ is $\Delta A(x)$.



(of course Δx is drawn much too large so as to understand where it is.)

Exercise 33

Using the drawing above, consider $f : x \mapsto 3x^2 + 1$. We would like to calculate the area between 1 and 3. For this we consider first the area up to 2 and its variation to $2 + \Delta x$.

(1) Write the formula for the variation of the area $\Delta A(2)$ or at least for upper and lower bounds to $\Delta A(2)$.

(2) Generalise to x and determine the equation of $A(x)$ – the area under f between 1 and x.

On the interval $[1, 3]$, the function is locally increasing – the derivative is positive, so if we zoom on it, locally it is increasing. Hence $f(2 + \Delta x) > f(2)$ for $\Delta x > 0$. The variation of the area is between $f(2) \cdot \Delta x$ and $f(2 + \Delta x) \cdot \Delta x$, hence

$$f(2) \cdot \Delta x < \Delta A(2) < f(2 + \Delta x)\Delta x$$

Then

$$f(2) < \frac{\Delta A(2)}{\Delta x} < f(2 + \Delta x)$$

and we conclude that $f(2) \simeq \frac{\Delta A(2)}{\Delta x}$ and the conclusion is the same for $\Delta x < 0$ (with $>$ instead of $<$)

Therefore $A'(2) = f(2)$ and in general we will have $A'(x) = f(x)$.

Using results of previous exercises, it is possible to check that $A(x) = x^3 + x$ but also $x^3 + x + k$ satisfies the requirement.

We know that $A(1) = 0$ (the area under f from 1 to 1...) hence $A(1) = 1^3 + 1 + k = 0 \Rightarrow k = -2$ and $A(3) = 3^3 + 1 - 2 = 26$.

Exercise 34

Calculate the area under $f : x \mapsto x^2$ and above the x-axis, between 2 and 5 i.e., $a = 2$ and $x = b$. Use that $A(2) = 0$

Answers to practice exercises

Answers to practice exercice 4, page 18

(1) 10

(2) -70

(3) 10

(4) 20

Answers to practice exercice 5, page 24

(1) $f'(x) = 3$

(2) $g'(x) = 4x - 1$

(3) $h'(x) = 15x^2 + 4x - 1$

(4) $k'(x) = 15x^2 + 4x + 3$

5

Differentiation Rules

Since observable numbers remain observable if we zoom further in, a property is not changed if the context is extended.

For the following rules, the proofs proceed by five steps:

(1) Definition of the derivative.

(2) Definition of the Δ.

(3) Definition of operations on functions.

(4) Expansion of $f(a + \Delta x)$ as $f(a) + \Delta f(a)$.

(5) Division by Δx.

(6) Algebra.

Exercise 35

Explain why if f is differentiable at a, then $\Delta f(a) \simeq 0$.

The previous property can be rewritten using the $y = f(x)$ notation, where y is a dependent variable. Then if y' exists, we have $y' \simeq \frac{\Delta y}{\Delta x}$ and $\Delta y \simeq 0$.

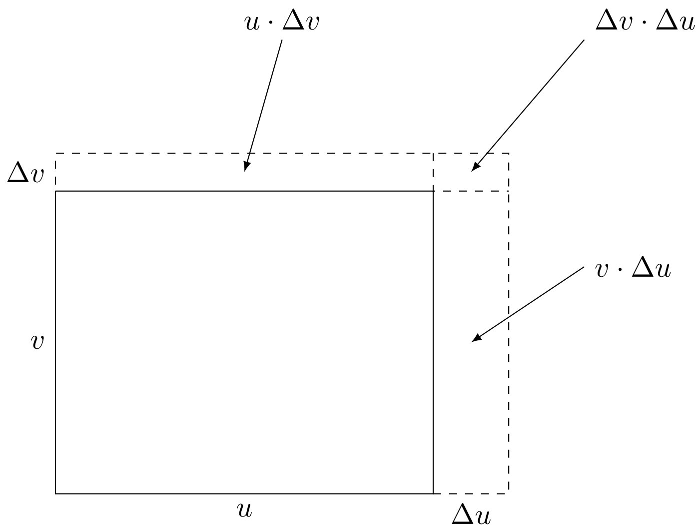
Product

> *Starting with linearity of the derivative leads to the common error $(uv)' = u'v'$. So we start with the less obvious ones to avoid this.*
> The notation $f(x) = u$ and other notations simplify the writing: it is a shift from function to dependent variable – which are similar concepts.

When two different functions are involved, it is common practice to write $f(x) = u$ and $g(x) = v$ then $\Delta f(x) = \Delta u$ and $\Delta g(x) = \Delta v$.

Consider the product $u \cdot v$ and its variation (a product $a \cdot b$ can be interpreted as the area of a rectangle with sides a and b).

When x varies to $x + \Delta x$, u varies to $u + \Delta u$ and v varies to $v + \Delta v$.



Then $u \cdot v$ varies to $v \cdot u + v \cdot \Delta u + \Delta v \cdot u + \Delta v \cdot \Delta u$ hence

$$\Delta(u \cdot v) = v \cdot \Delta u + \Delta v \cdot u + \Delta v \cdot \Delta u$$

Exercise 36
Divide the expression above by Δx and justify that $\frac{\Delta u \cdot \Delta v}{\Delta x} \simeq 0$ to prove

$$\frac{\Delta u \cdot \Delta v}{\Delta x} = \frac{\Delta u}{\Delta x} \cdot \Delta v \simeq u' \cdot \Delta v$$

Since $\Delta v \simeq 0$ we have

$$u' \cdot \Delta v \simeq 0$$

Theorem 8
Let u and v be two differentiable functions, then

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$\frac{\Delta(u \cdot v)}{\Delta x} = \frac{\Delta u}{\Delta x} \cdot v + u \cdot \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \Delta v \simeq u' \cdot v + u \cdot v'$$

This theorem can also be written:

Let f and g be two real functions differentiable at a. Then the function $f \cdot g$ is differentiable at a and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a).$$

Exercise 37
Using the derivatives of $f : x \mapsto x^2$ and $g : x \mapsto x^3$, calculate the derivative of $h : x \mapsto x^5$ $(= x^2 \cdot x^3)$.

Exercise 38

Let c be a constant, considered as a constant function. What is Δc? and use this to conclude that

Theorem 9

Let c be a constant. Then

$$c' = 0$$

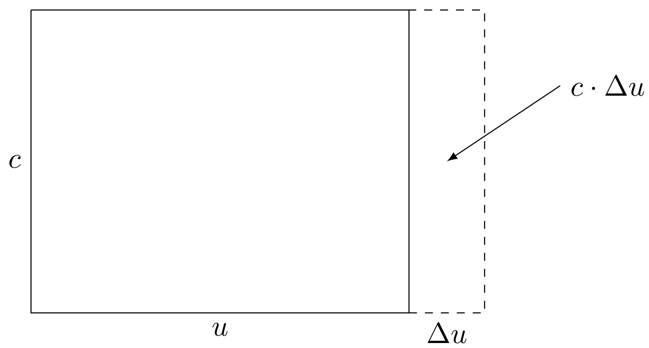
This theorem can also be written:

Let $c \in \mathbb{R}$ and $f : x \mapsto c$, for $x \in \mathbb{R}$

$$f'(x) = 0.$$

Consider the product $c \cdot u$ for constant c and differentiable function u, then when x varies to $x + \Delta x$ the product $c \cdot u$ varies $to c \cdot u$ to $c \cdot u + c \cdot \Delta u$, hence

$$\Delta(c \cdot u) = c \cdot \Delta u$$



Exercise 39

Divide the expression above by Δx to prove

Theorem 10

Let c be a constant and u a differentiable function. Then

$$(c \cdot u)' = c \cdot u'$$

$$\frac{c\Delta u}{\Delta x} = c \cdot \frac{\Delta u}{\Delta x} \simeq c \cdot u'$$

This theorem can also be written:

Let $c \in \mathbb{R}$ and f be a real function differentiable at a. Then the function $c \cdot f$ is differentiable at a and

$$(c \cdot f)'(a) = c \cdot f'(a).$$

The following theorem expresses a property for all natural numbers:

Theorem 11

$$(x^n)' = n \cdot x^{n-1}.$$

It is of course impossible to prove all cases. We prove by induction.
If

(1) The property holds for $n = 0$ (or $n = 1$),

(2) Assuming the property holds for n greater than 0 (or 1), we can prove that it also holds for $n + 1$,

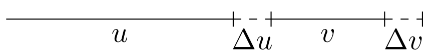
then the property holds for all n.

A proof that this method of proof is valid can be given by contradiction. Assume (1) and (2) have been checked but that there is a value m such that the property does not hold for m. Then $m > 1$ since that has been proven to be true. Let n be the smallest number such that the property does not hold. (This number is not zero because of (1).) Then the property holds for $n - 1$. But by (2), this proves that the property holds for n: a contradiction. So there can be no number for which the property does not hold.

A function such as $f : x \mapsto (x^3+2x)^4$ can be decomposed as a composition of $f_1 : x \mapsto x^3+2x$ and $f_2 : x \mapsto x^4$. Then $f = f_2 \circ f_1$.

Sum and Difference

Consider the sum. When x varies to $x + \Delta x$, u varies to $u + \Delta u$ and v varies to $v + \Delta v$.



Then

$$\Delta(u + v) = \Delta u + \Delta v$$

Exercise 40
Divide the expression above to prove:

Theorem 12
Let u and v be differentiable functions. Then

$$(u + v)' = u' + v'$$

$$\frac{\Delta u + \Delta v}{\Delta x} \simeq u' + v'$$

This theorem can also be written:

Let f and g be real functions differentiable at a. Then the function $f + g$ is differentiable at a and

$$(f + g)'(a) = f'(a) + g'(a).$$

Exercise 41
Find the derivatives of $h : x \mapsto x^3 + x^2$ and $k : x \mapsto 5x^3 - 7x^2$.

Composition

Theorem 13 (Chain Rulle)
Let u by a differentiable function of v and v a differentiable function of x. Then

$$(u \circ v)' = u' \cdot v'$$

Exercise 42
Prove the chain rule.

If u' exists, we have (as usual)
$$u' \simeq \frac{\Delta u}{\Delta x}$$
where u depends on v
If $\Delta v \neq 0$, then
$$u' \simeq \frac{\Delta u}{\Delta x} = \frac{\Delta u}{\Delta v} \cdot \frac{\Delta v}{\Delta x} \simeq u' \cdot v'$$

Exercise 43
Prove that this formula holds also if $\Delta v = 0$.

If $\Delta v = 0$ then $v' = 0$, so $f'(v) \cdot v' = 0$. But since v has no variation, u has no variation, so $u' = 0$ and the result also holds.

This theorem can also be written:

Let f and g be real functions such that g is differentiable at a and f is differentiable at $g(a)$. The the function $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Exercise 44
Give the derivatives of the following functions:

(1) $f : x \mapsto (x^3 + 2x)^4$

(2) $g : x \mapsto (5x^3 + 3x^2)^{13}$

Exercise 45

Use $(\sqrt{x})^2 = x$ and theorem 13 to find the derivative of $y = \sqrt{x}$ (for $x > 0$) – assuming it exists.

Exercise 46

Give the derivatives of the following functions:

(1) $f : x \mapsto (\sqrt{x} + 1)^4$

(2) $g : x \mapsto \sqrt{5x^3 + 3x^2}$

(3) $h : x \mapsto \sqrt{x^2}$

Exercise 47

Find the derivatives of the following:

(1) $y = \sqrt{3x^3 + 2x + 1}$

(2) $y = (x^2 + 3)^5$

(3) $y = (ax + b)^n$

(4) $y = \sqrt{x^3 + 1}$

Exercise 48

Use the definition of the derivative to find $f'(x)$ for $f : x \mapsto \frac{1}{x}$

Exercise 49

Use the previous exercise and the chain rule to find the derivative of $\frac{1}{f(x)}$ assuming $f(x) \neq 0$ and $f'(x)$ exists.

Write $f(x) = u$. Since $\left(\frac{1}{x}\right)' = -\frac{1}{x2}$ (by previous exercise) we have $\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$

Quotient

Exercise 50

Use all previous results to prove:

Theorem 14

Let u and v be differentiable functions with $v \neq 0$, then

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$\frac{u}{v} = u \cdot \frac{1}{v}$ hence

$$\left(\frac{u}{v}\right)' = u' \cdot \frac{1}{v} - u \cdot \frac{v'}{v^2} = \frac{u' \cdot v - u \cdot v'}{v^2}$$

This proof is nice because it uses the chain rule and therefore stresses its importance.

Or more classical:

$$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{\Delta u \cdot v - u \cdot \Delta v}{v^2 + v \cdot \Delta v}$$

$$\frac{\Delta\left(\frac{u}{v}\right)}{\Delta x} = \frac{\dfrac{\Delta u}{\Delta x} \cdot v - u \cdot \dfrac{\Delta v}{\Delta x}}{v^2 + v \cdot \Delta v} \simeq \frac{u' \cdot v - u \cdot v'}{v^2}$$

Also written:

Let f and g be two real functions differentiable at a and $g(a) \neq 0$. Then the function $\dfrac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g^2(a)}.$$

Exercise 51

Show that for $m \in \mathbb{Z}$

$$(x^m)' = m \cdot x^{m-1}.$$

Exercise 52

Find the slope of $x \mapsto \dfrac{x^2 - 2x}{x^3 + x^2}$ at $x = 1$.

Exercise 53

Find the derivative of

$$f : x \mapsto \frac{x}{x^2 + 1}$$

Practice exercise 6 Answer page 40

Differentiate the following for general x:

(1) $f : x \mapsto 5x^4 + x^3 - 2x^2 + 25$

(2) $g : x \mapsto 5\sqrt{3}\, x^2 - 100$

(3) $h : x \mapsto \dfrac{x^2 + 2x - 1}{x^3 - 5}$

(4) $j : x \mapsto 5x^4 + \dfrac{1}{3x^2 - 2x + \pi}$

(5) $k : x \mapsto (5x + 2) \cdot \dfrac{1}{5x + 2}$

(6) $l : x \mapsto \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4}$

(7) $m : x \mapsto \dfrac{1 + x}{1 + \frac{1+x}{x^2}}$

Practice exercise 7 Answer page 40

Sketch the curve of $y = -(x-3)(x+1)(x-1)$.

Practice exercise 8 Answer page 40

Let $y = \dfrac{10x}{x^2+1}$. Sketch the curve and give the equation of the line tangent to the curve at $x = 3$.

Practice exercise 9 Answer page 41

Consider each of the following as a function f, find the corresponding derivative function f'.

(1) $x^3 + x^2 + 2x - 4$

(2) $-x^3 + 2x^2 - 2x + 1$

(3) $\frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$

(4) $\frac{1}{3}(x-2)^3$

(5) $\dfrac{x^2}{x+2}$

(6) $x - 1 + \dfrac{9}{x+1}$

(7) $\dfrac{4x^2+4x+5}{4x+2}$

(8) $\dfrac{-x^2-2x-1}{x+3}$

(9) $|x-2|$

(10) $\dfrac{x^2}{|x|+2}$

(11) $x + 2 - \dfrac{1}{x+1}$

(12) $|x^3 - 6x^2 + 11x - 6|$

Exercise 54

Find the derivative of the following functions. Since they are piecewise defined, the answer will be in 3 parts – one special point is the meeting point for bothe rules.

(1)

$$f : x \mapsto \begin{cases} x^2 & \text{if } x \geq 1 \\ 2x - 1 & \text{if } x < 1 \end{cases}$$

(2)

$$g : x \mapsto \begin{cases} x^2 & \text{if } x > 2 \\ x + 2 & \text{if } x \leq 2 \end{cases}$$

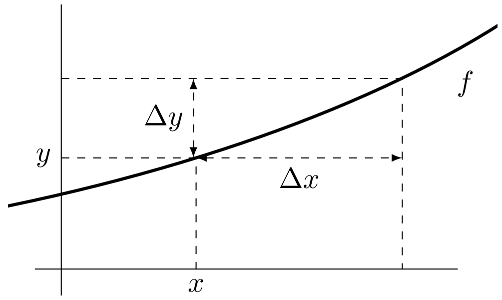
(3)

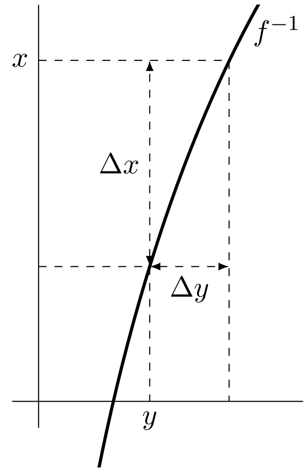
$$h : x \mapsto \begin{cases} x^2 & \text{if } x \geq 3 \\ 2x & \text{if } x < 3 \end{cases}$$

Let f be a function. Recall that the inverse function of f, if it exists, is written f^{-1} and is such that $f^{-1}(f(x)) = x$ amd if we write $f(x) = y$ then we also have $f(f^{-1}(y)) = y$.

⚠ $f^{-1}(x)$ is <u>not</u> $\dfrac{1}{f(x)}$.

A function has an inverse if the image of its curve by a symmetry through the $y = x$ axis is the curve of a function.





The slope of the tangent of the inverse is the reciprocal of the slope of the original tangent:

$$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}}$$

Theorem 15 (Derivative of the Inverse)
If $f : I \to J$ is a function, differentiable on I and has an inverse f^{-1}, and $f'(a) \neq 0$ then this inverse is differentiable at $b = f(a) \in J$ and

$$\frac{\Delta f^{-1}(b)}{\Delta y} = \frac{1}{f'(a)}.$$

In general form:

$$\frac{\Delta f^{-1}(y)}{\Delta y} = \frac{1}{f'(x)}$$

Exercise 55
Find the derivative of $y = x^{\frac{1}{n}}$.

This shows that the rule in exercise 11 holds also for rational n.

Exercise 56
Use $\mid x \mid = \sqrt{x^2}$ to find an expression for the derivative of $\mid x \mid$.

Theorem 16 (Derivative at a maximum or a minimum.)
Let f be a real function defined on an open interval $]a; b[$ differentiable at $c \in]a; b[$.
If $f(c)$ is a local maximum (or a local minimum) then $f'(c) = 0$.

Exercise 57

Prove theorem 16. (Hint, consider that the derivative must be ultraclose to $\dfrac{\Delta f(c)}{\Delta x}$ whether Δx is positive or negative.)

Assume $f'(a)$ exists and that $\langle a, f(a)\rangle$ is a local maximum. (the same proof holds for a minimum. Then $f(a) \geq f(a + \Delta x) \Rightarrow f(a + \Delta x) - f(a) \leq 0$.
Let Δx be positive, then $\frac{f(a+\Delta x)-f(a)}{\Delta x} \leq 0 \simeq f'(a)$
Let Δx be negative, then $\frac{f(a+\Delta x)-f(a)}{\Delta x} \geq 0 \simeq f'(a)$
The only observable number which is ultraclose to positive and negative values is 0.

Exercise 58

Find the derivative of $f : x \mapsto x^3$ at $x = 0$ to see that the converse of theorem 16 does not hold.

Optimisation and Other Problems

Exercise 59

A $1l$ milk pack is made of cardboard. Its sides can only be rectangles. The height is twice one of the other two dimensions. The area of the outside of the pack must be minimal.

What are the dimensions of the pack?

Exercise 60

Imagine you want to protect a part of a rectangular garden against a wall. You have 100m of fence. (No fence is needed against the wall.)

What is the biggest area that you can protect?

Exercise 61

A cylindrical jar has a volume defined by its radius and its height. If it contains one litre (1dm^3), what are the dimensions that will make it have the least outside area?

Exercise 62

Find the length and width of the rectangle inscribed within the ellipse given by the formula $4x^2 + y^2 = 16$ (sides parallel to the coordinate axes) such that its area is maximal.

Exercise 63

Let $\mathcal{P}$ be the parabola given by $x \mapsto x^2$ and A be the point $\langle 0; 5\rangle$.

Find the point(s) on the parabola $\mathcal{P}$ such that its (their) distance to A is minimal.

Exercise 64

(1) Find the slope of the curve given by $y = 5x^3 - 25x^2$ at $x = 3.5$.

Equivalent statement: compute $f'(x)\Big|_{x=3.5}$

(2) Find the equation of the line tangent to the curve at $x = 1$.

Exercise 65

(1) For $f : x \mapsto x^2 + 5$ and the point $A\langle 0; 0\rangle$, what is the equation of the line passing through A, and tangent to f? Is it unique?

(2) If $g : x \mapsto ax^2 + b$, what values do a and b take to make $g(x)$ tangent to $t : x \mapsto 3x - 2$ at $x = 5$? What are the coordinates of the contact point?

Summary

- $c' = 0$
- $(c \cdot u)' = c \cdot u'$
- $(u + v)' = u' + v'$
- $(u \cdot v)' = u' \cdot v + u \cdot v'$
- $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$
- $(u \circ v)' = u' \cdot v'$ (in this case, u depends on v which depends on x).

Answers to practice exercises

Answers to practice exercice 6, page 35

(1) $f'(x) = 20x^3 + 3x^2 - 4x$

(2) $g'(x) = 10\sqrt{3}x$

(3) $h'(x) = -\dfrac{x^4 + 4x^3 - 3x^2 + 10x + 10}{(x^3 - 5)^2}$

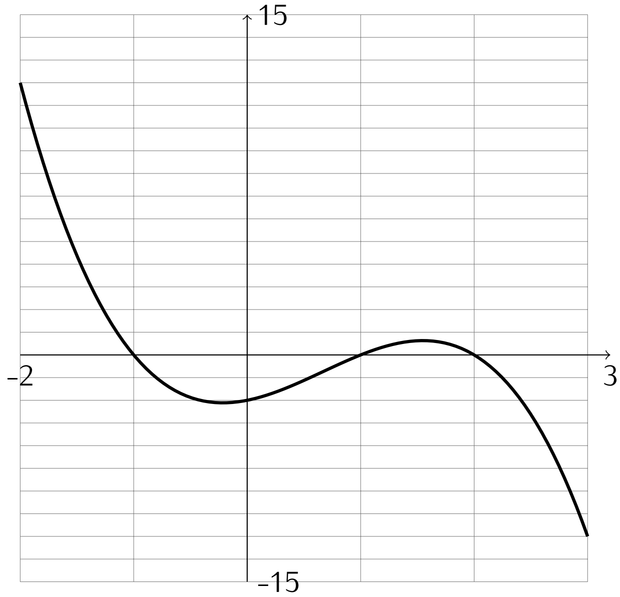
(4) $j'(x) = 20x^3 - \dfrac{6x - 2}{(3x^2 - 2x + \pi)^2}$

(5) $k'(x) = 0$

(6) $l'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} - \dfrac{3}{x^4} - \dfrac{4}{x^5}$

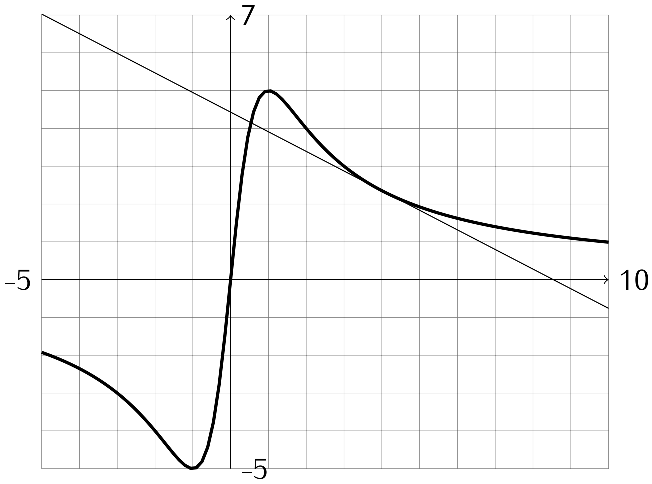
(7) $m'(x) = \dfrac{(x^2 + x + 1)(3x^2 + 2x) - (x^3 + x^2)(2x + 1)}{(x^2 + x + 1)^2} = \dfrac{x(x^3 + 2x^2 + 4x + 2)}{(x^2 + x + 1)^2}$

Answers to practice exercice 7, page 36



Answers to practice exercice 8, page 36

Tangent line is $y = -\dfrac{4}{5}x + \dfrac{27}{5}$



Answers to practice exercice 9, page 36

(1) $3x^2 + 2x + 2$

(2) $-3x^2 + 4x - 2$

(3) $x^2 - 5x + 6$

(4) $(x - 2)^2$

(5) $\dfrac{x(x + 4)}{(x + 2)^2}$

(6) $\dfrac{x^2 + 2x - 8}{(x + 1)^2}$

(7) $\dfrac{4x^2 + 4x - 3}{(2x + 1)^2}$

(8) $-\dfrac{x^2 + 6x + 5}{(x + 3)^2}$

(9) $\begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \\ \text{not differentiable} & \text{if } x = 2 \end{cases}$

(10) $\begin{cases} \dfrac{x(x + 4)}{(x + 2)^2} & \text{if } x \geq 0 \\ \\ \dfrac{-x(x - 4)}{x - 2)^2} & \text{if } x \leq 0 \end{cases}$

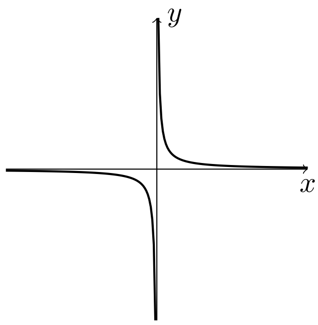
(11) $\dfrac{x^2 + 2x + 2}{(x + 1)^2}$

(12) $\begin{cases} 3x^2 - 12x + 11 & \text{if } x \in]1; 2[\cup]3; \infty[\\ -3x^2 + 12x - 11 & \text{if } x \in] - \infty; 1[\cup]2; 3[\\ \text{not differentiable} & \text{if } x \in \{1; 2; 3\} \end{cases}$

6
Asymptotes

Exercise 66

Consider the real function $f : x \mapsto \frac{1}{x}$.



(1) What is the domain of this function? Specify the context.

(2) What happens to the curve close to the vertical axis i.e., for values of x close to 0? Consider ultrasmall values of x.

(3) What happens to the curve close to the horizontal axis? i.e., for very large values of x? Consider ultralarge values of x (positive or negative).

(4) Draw this function for a horizontal range of $[-100; 100]$ and a vertical range of $[-100; 100]$.

Informally: For a given function f, a straight line is **an asymptote** of the function f if it is ultraclose to the function when either

- x is ultralarge (horizontal or oblique asymptote).
- y (or $f(x)$) is ultralarge (vertical asymptote).

Definition 10
A real function f ***has a vertical asymptote at*** *$x = a$ if $f(x)$ is positive or negative ultralarge for $x \simeq a$, x being less than a or x being greater than a.*
If it is the case for x greater than a, we write

$$x \simeq a_+ \Rightarrow f(x) \text{ is ultralarge}$$

If it is the case for x less than a, we write

$$x \simeq a_- \Rightarrow f(x) \text{ is ultralarge}$$

Example: The function $f : x \mapsto 1/x$ has a vertical asymptote at 0. The only parameter of the function is 1, always observable. If Δx is a positive ultrasmall number then $f(\Delta x)$ is positive ultralarge. Hence

$$\frac{1}{\Delta x} \text{ is ultralarge}$$

Exercise 67
Show that $f : x \mapsto \frac{1}{x-2}$ has a vertical asymptote at $x = 2$.
Give the domain of f.

Exercise 68
Show that

$$g : x \mapsto \begin{cases} \dfrac{1}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

has a vertical asymptote at $x = 2$.
Give the domain of g.

Exercise 69
Show that $h : x \mapsto (|x|)'$ has no vertical asymptote at $x = 0$.
Give the domain of h.

From the previous exercises we can see that there is no immediate link between the fact that values are missing in a domain and the existence of asymptotes.

	values missing in domain	asymptote
f	yes	yes
g	no	yes
h	yes	no

Definition 11
A real function f has a ***horizontal asymptote on the right*** *(resp. on the left) if there is an observable number L such that*

$$x \text{ ultralarge positive (resp. negative) } \Rightarrow f(x) \simeq L.$$

Example: Consider

$$\frac{x^2 - 3x + 1}{x^2 + 1} \text{ for ultralarge } x.$$

This means: consider the fraction for an ultralarge value of x.

The function $f : x \mapsto \dfrac{x^2 - 3x + 1}{x^2 + 1}$ is defined on $\mathbb{R}$. $1, 2$ and 3 are always observable. Let x be ultralarge. Then

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 + 1} = \frac{x^2(2 - \frac{3}{x} + \frac{1}{x^2})}{x^2(1 + \frac{1}{x^2})} = \frac{2 - \overbrace{\frac{3}{x}}^{\simeq 0} + \overbrace{\frac{1}{x^2}}^{\simeq 0}}{1 + \underbrace{\frac{1}{x^2}}_{\simeq 0}} \simeq \frac{2}{1} = 2,$$

hence f has a horizontal asymptote $y = 2$.

Exercise 70

Show that $f : x \mapsto \dfrac{x}{x^2 + 1}$ has a horizontal asymptote at $y = 0$.

Find the value of x for which f crosses its horizontal asymptote.

We now define the oblique asymptote

Definition 12

*A real function f has an **oblique asymptote at** on the right (resp. on the left) if there exist observable numbers a, b such that, if x is ultralarge positive (resp. negative), then*

$$f(x) - (ax + b) \simeq 0$$

The line $y = ax + b$ is the **oblique asymptote of** f

The existence of an oblique asymptote is a property of f hence the context is f.

This is equivalent to saying that $f(x) \simeq ax + b$ whenever x is ultralarge.

Example: Consider

$$f : x \mapsto \frac{x^3 + 2x^2 + x - 1}{x^2 + 1}$$

defined on $\mathbb{R}$. Using long division we have

$$f(x) = x + 2 - \frac{3}{x^2 + 1}.$$

Let x be ultralarge. We have

$$f(x) - (x + 2) = \frac{-3}{x^2 + 1} \simeq 0,$$

because $x^2 + 1$ is ultralarge. Hence f has an oblique asymptote at $y = x + 2$, i.e., $a = 1$ and $b = 2$.

Exercise 71

Find the asymptotes (if any) of

(1) $f : x \mapsto \dfrac{x}{2x^2+1}$

(2) $g : x \mapsto \dfrac{2x^2+1}{x}$

(3) $h : x \mapsto \dfrac{x^3+2}{2x^2-1}$

(4) $i : x \mapsto \dfrac{x^2+2x+1}{x+1}$

(5) $j : x \mapsto \dfrac{3x^3+2x^2-x+12}{x^2+8}$

For functions which are not rational functions, where the polynomial long division does not apply, we have the following:

Theorem 17

Let f be a real function and let a and b be observable (context is f). Then f has an oblique asymptote at $y = ax + b$ on the right (resp. on the left) if and only if, for ultralarge positive (resp. negative) x, there are observable a and b such that

$$\frac{f(x)}{x} \simeq a \quad \textit{and} \quad (f(x) - ax) \simeq b.$$

Remark: If $a = 0$ the line $y = ax + b$ becomes $y = b$ i.e., a horizontal asymptote.

Since the asymptote is a property of the function, the context is given by f but not by x.
If f has an oblique asymptote $y = ax + b$ then for ultralarge x, we have $f(x) \simeq ax + b$.
Divide by x:

$$\frac{f(x)}{x} \simeq a + \underbrace{\frac{b}{x}}_{\simeq 0} \simeq a$$

and $f(x) - ax \simeq b$.

Conversely, assume that for ultralarge x, $\frac{f(x)}{x} \simeq a$ and $f(x) - ax \simeq b$, then it is immediate that for ultralarge x, $f(x) \simeq ax + b$.

Exercise 72

Prove the previous theorem.

Example: Consider $f : x \mapsto \sqrt{x^2+1}$ defined on $\mathbb{R}$. Let x be positive ultralarge. Then

$$\frac{f(x)}{x} = \frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{x^2(1+1/x^2)}}{x} = \frac{|x|\overbrace{\sqrt{1+1/x^2}}^{\simeq 1}}{x} \simeq \begin{cases} 1 & \text{it } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Moreover:

$$f(x) - x = \sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1} - x)\cdot(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} = \frac{1}{\sqrt{x^2+1} + x} \simeq 0.$$

Hence f has an oblique asymptote at $y = x$ on theb right.

On the left, the function has an oblique asymptote at $y = -x$.

Exercise 73

Find the asymptotes at infinity (if any) of

(1) $i : x \mapsto x^{\frac{3}{2}}$

Practice exercise 10 Answer page 52

Find all asymptotes of the following functions.

(1) $f_1 : x \mapsto \dfrac{x^2 - x}{x - 1}$

(2) $f_2 : x \mapsto \dfrac{4x^3 + 2x^2 - 5}{3x^3 - 4x^2}$

(3) $f_3 : x \mapsto \sqrt{x^2 + x}$

(4) $f_4 : x \mapsto \dfrac{\sqrt{x^5 + x}}{\sqrt{3x^5 - x}}$

(5) $f_7 : x \mapsto \dfrac{x^1 0}{x^1 0 + 1}$

Theorem 18 (Rule of de l'Hospital for $0/0$)
Let f and g be differentiable functions at a. Suppose that $f(a) = g(a) = 0$, but that $g'(a) \neq 0$. Then

$$\frac{f(a + \Delta x)}{g(a + \Delta x)} \simeq \frac{f'(a)}{g'(a)}$$

(provided $f'(a)$ and $g'(a)$ exist).

Exercise 74

Prove theorem 18.

Write $f(a + \Delta x) = u + \Delta$ and $g(a + \Delta x) = v + \Delta v$, then for $x \simeq a$ we write

$$\frac{u + \Delta u}{v + \Delta v} \underbrace{=}_{u=0,v=0} \frac{\Delta u}{\Delta v} = \frac{u'\Delta x + \varepsilon\Delta x}{v'\Delta x + \delta\Delta x} = \frac{u' + \varepsilon}{v' + \delta} \simeq \frac{u'}{v'}$$

The rule of de l'Hospital also holds for the case where f and g need not be defined at $x = a$, but

$$x \simeq a \Rightarrow f(x) \simeq 0 \text{ and } g(x) \simeq 0$$

(if $g'(x) \not\simeq 0$) case $\frac{\text{ultrasmall}}{\text{ultrasmall}}$ and also for the case $\frac{\text{ultralarge}}{\text{ultralarge}}$.

The proof of this general case goes beyond classroom work. It requires considering $x \simeq a$ then $f'(x)$ which requires $\Delta x \simeq 0$ in the extended context of f and x, this means working with three levels. See the book.

Exercise 75

Assuming the rule of de l'Hospital holds for the case $\frac{\text{ultrasmall}}{\text{ultrasmall}}$, show that it holds for the case $\frac{\text{ultralarge}}{\text{ultralarge}}$

Assume that $f(x) = u$ and $g(x) = v$ are ultralarge. Then $\frac{u}{v} = \frac{\frac{1}{v}}{\frac{1}{u}}$ which is $\frac{\text{ultrasmall}}{\text{ultrasmall}}$ so

$$\frac{u}{v} \simeq \frac{\left(\frac{1}{v}\right)'}{\left(\frac{1}{u}\right)'} = \frac{-\frac{v'}{v^2}}{-\frac{u'}{u^2}} = \frac{v'}{u'}\frac{u^2}{v^2}$$

Hence

$$\frac{u}{v} \simeq \frac{v'}{u'}\frac{u^2}{v^2}$$

which leads to

$$\frac{v}{u} \simeq \frac{v'}{u'}$$

Exercise 76

Evaluate using de L'Hospital's rule.

(1) $\dfrac{1/t - 1}{t^2 - 2t + 1}$ for $t \simeq 1$ (with $(t > 1)$).

(2) $\dfrac{\sqrt{x} - 1}{\sqrt[3]{x} - 1}$ for $x \simeq 1$.

(3) $\dfrac{x^2}{\sqrt{2x+1} - 1}$ for $x \simeq 0$.

(4) $\dfrac{2 + 1/t}{3 - 2/t}$ for $t \simeq 0$.

(5) $\dfrac{x + 5 - 2x^{-1} - x^{-3}}{3x + 12 - x^{-2}}$ for ultralarge x

(6) $\left(t + \dfrac{1}{t}\right)((4 - t)^{3/2} - 8)$ for $t \simeq 0$.

(7) $\dfrac{u + u^{-1}}{1 + \sqrt{1 - u}}$ for ultralarge u.

Practice exercise 11 Answer page 52
Evaluate using de L'Hospital's rule.

(1) $\dfrac{\sqrt{9+x}-3}{x}$ for $x \simeq 0$

(2) $\dfrac{2-\sqrt{x+2}}{4-x^2}$ for $x \simeq 2$

(3) $\dfrac{\sqrt{u+1}+\sqrt{u-1}}{u}$ for ultralarge u

(4) $\dfrac{(1-x)^{1/4}-1}{x}$ for $x \simeq 0$

(5) $\left(\dfrac{1}{t}+\dfrac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)$ for $x \simeq 0_+$

(6) $\dfrac{(u-1)^3}{u^{-1}-u^2+3u-3}$ for $u \simeq 1$

(7) $\dfrac{1+5/\sqrt{u}}{2+1/\sqrt{u}}$ for $u \simeq 0_+$

(8) $\dfrac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$ for ultralarge x

(9) $\dfrac{1-t/(t-1)}{1-\sqrt{t/(t-1)}}$ for ultralarge t

7
Curve Sketching

Curve sketching needs the following steps:

- Find the domain.
- Find the roots and the intercept (if any).
- Find the asymptotes (if any).
- Find the derivative (if any).
- Find the roots of the derivative (if any).
- Determine the maximums and minimums.
- Put all these values in a table.
- Draw arrows which indicate the general direction of the function:
- Use this information to choose a convenient scale.
- Sketch the function.

Practice exercise 12 Answer page 52

(1) $f_1 : x \mapsto \dfrac{x^2}{x+2}$

(2) $f_2 : x \mapsto x - 1 + \dfrac{9}{x+1}$

(3) $f_3 : x \mapsto \dfrac{-x^2 - 2x - 1}{x+3}$

(4) $f_4 : x \mapsto x + 3 + \dfrac{1}{2x+1}$

(5) $f_5 : x \mapsto \dfrac{x^2 - 4x + 6}{(x-2)^2}$

(6) $f_6 : x \mapsto \dfrac{2x^2 - 3}{x^2 - 1}$

(7) $f_7 : x \mapsto \dfrac{x^2 + 3x - 4}{x^2 - x - 2}$

(8) $f_8 : x \mapsto \dfrac{x^3 + 2}{2x}$

(9) $f_9 : x \mapsto \dfrac{x^3 - 1}{x^2}$

(10) $f_{10} : x \mapsto \dfrac{2x - 1}{\sqrt{x^2 + 2}}$

(11) $f_{11} : x \mapsto \dfrac{\sqrt{x^2 + 1}}{x + 1}$

(12) $f_{12} : x \mapsto \dfrac{\sqrt{x^2 - 4x + 3}}{x + 1}$

Answers to practice exercises

Answers to practice exercice 10, page 47

Vertical asymptote of the form $x = c$, horizontal asymptote of the form $y = b$, oblique asymptote of the form $y = ax + b$.

(1) $y = x$

(2) $y = 1$, $x = 0$, $x = 4/3$

(3) $\begin{cases} y = x & \text{if } x > 0 \\ y = -x & \text{if } x < 0 \end{cases}$

(4) $y = \sqrt{1/3}$, $x = \sqrt[4]{1/3}$

(5) $\begin{cases} y = 0 & \text{if } x < 0 \\ y = 1 & \text{if } x > 0 \end{cases}$

Answers to practice exercice 11, page 49

(1) $1/6$

(2) $1/16$

(3) 0

(4) $-1/4$

(5) $1/2$

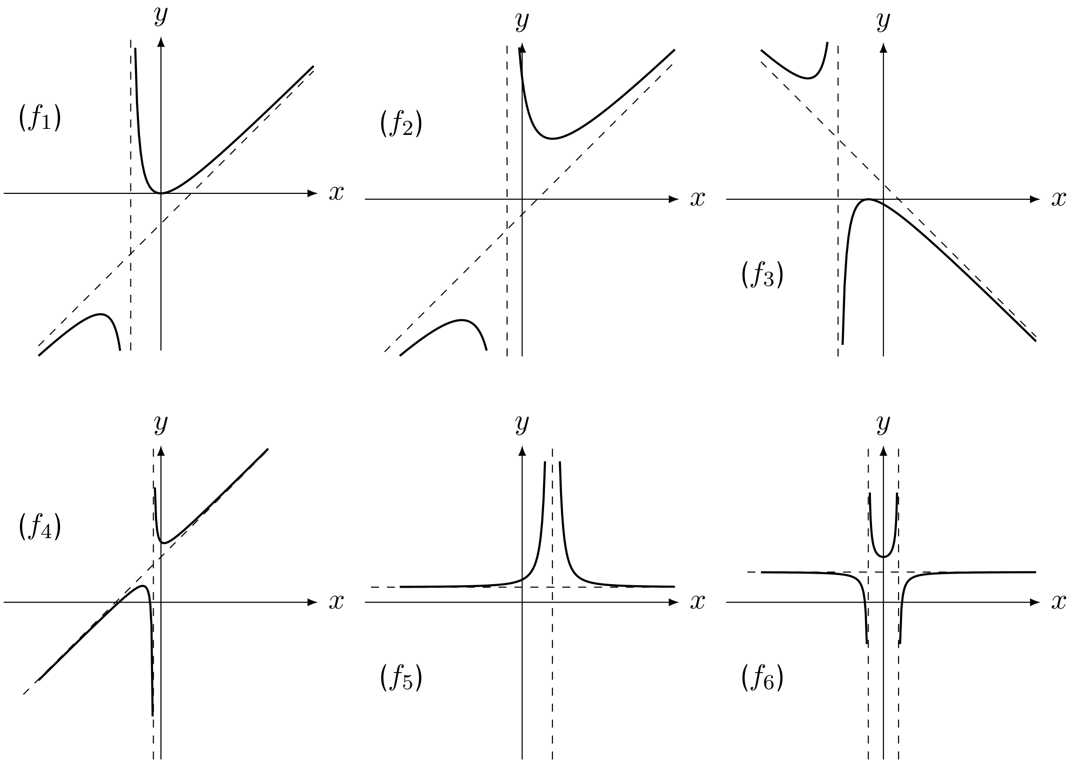
(6) -1

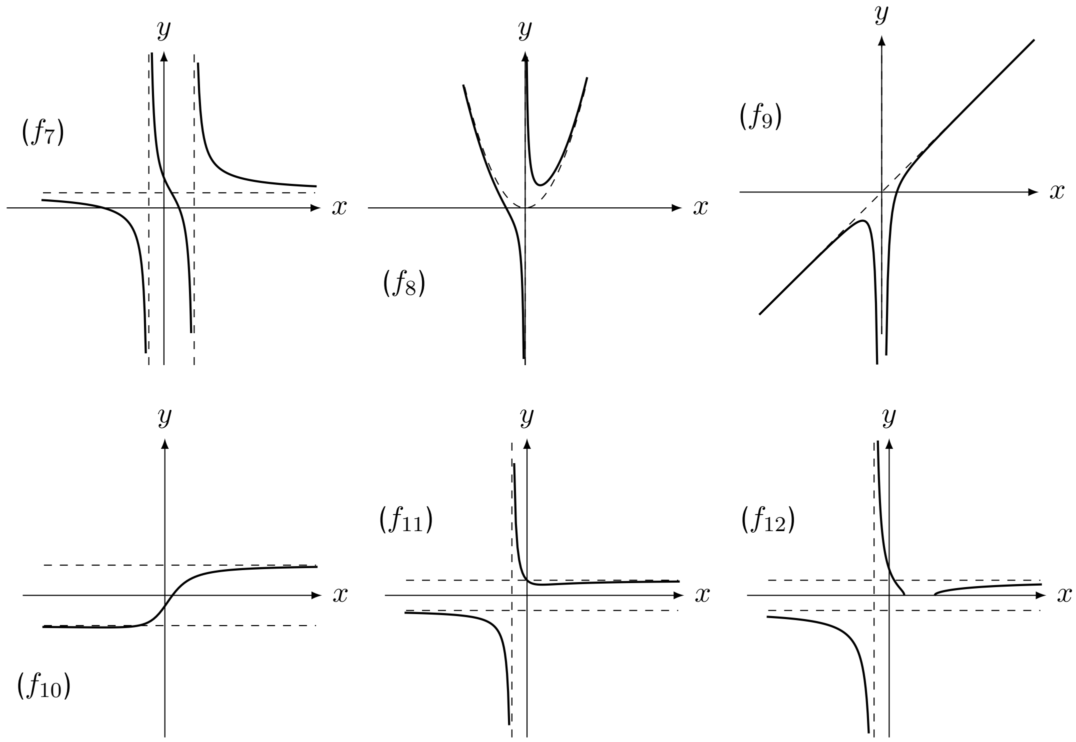
(7) 5

(8) ultralarge

(9) 2

Answers to practice exercice 12, page 51




(f_7)
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x
y

Analysis
with ultrasmall numbers

Standard Level

PART II

8
Continuity and Differentiability

Theorem 19 (Intermediate Value theorem)
Let f be a real function continuous on $[a;b]$. Let d be a real number between $f(a)$ and $f(b)$. Then there exists c in $[a;b]$ such that $f(c) = d$.

This theorem does not tell us how to find the root or the value c such that $f(c) = d$. It only asserts the *existence* of such a number. For specific functions where we can calculate explicitly the roots this theorem is not really necessary but, when proving theorems about continuous functions in general, it is the only way to know that there is a root.

Exercise 77
Give an example of a function f discontinuous on $[a;b]$ with $f(a) < 0$ and $f(b) > 0$ such that there is no c in the interval $[a;b]$ such that $f(c) = 0$.

For standard level, I do not ask for the proof, but I may do it in class.

Let N be an ultralarge integer, and $\Delta x = \frac{b-a}{N} \simeq 0$ and $x_k = a + k \cdot \Delta x$.
Let x_j be the first element of the partition $\{a, x_1, x_2, \ldots, x_N = b\}$ such that $f(x_j) < 0$ and $f(x_{j+1}) \geq 0$.
Since $a \leq x_j \leq b$, then x_j has an observable neighbour c, so $x_j \simeq c$ and $x_{j+1} \simeq c$. By closure $f(c)$ is observable with $f(c) \simeq f(x_j) < 0$ and $f(c) \simeq f(x_{j+1}) \geq 0$, hence $f(c) = 0$.

I usually give the example of $x \mapsto x^2 - 2$ as $f : \mathbb{Q} \to \mathbb{Q}$ to show that this theorem is the link between continuity and the fundamental characterisation of what real numbers are.

Definition 13
*A function is **smooth** if it is differentiable and its derivative is continuous.*

Almost all functions encountered so far are smooth.

Definition 14
*A function has **maximum** (respectively **minimum**) on an interval I if there is a $c \in I$ such that for any $x \in I$ we have $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$).*
*If a point is either a maximum or a minimum, it is an **extremum**.*

Theorem 20 (Extreme value)
Let f be a function continuous on $[a;b]$ smooth on $]a,b[$. Then it has a (local) maximum and a (local) minimum on $[a;b]$.

Not done in class. For a proof, see book or higher level handout.

Theorem 21 (Rolle)
Let f be a real function continuous on $[a;b]$ and smooth on $]a;b[$. If $f(a) = f(b)$, then there is a $c \in]a;b[$ such that

$$f'(c) = 0.$$

Exercise 78

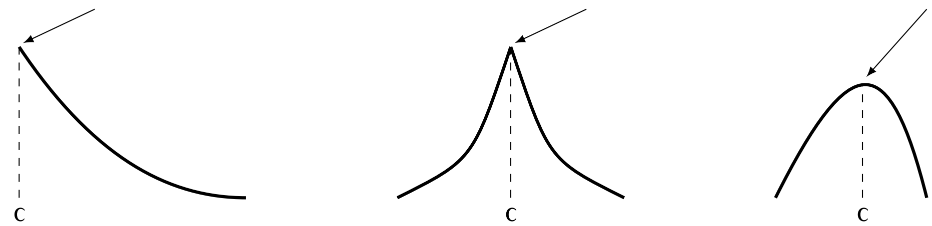
Prove Rolle's theorem.

Theorem 22 (Critical Point Theorem)
Let f be a function smooth on I and suppose that c is a point in I and f has either a maximum or a minimum at c. Then one of the following three things must happen:

(1) c is an end point of I.

(2) $f'(c)$ is undefined.

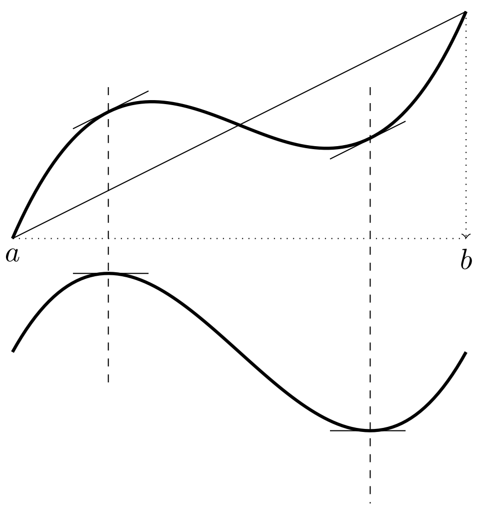
(3) $f'(c) = 0$



Theorem 23 (Mean Value)
Let f be a real function continuous on $[a;b]$ and smooth on $]a;b[$. Then there is a $c \in]a;b[$ such that

$$f(b) - f(a) = f'(c) \cdot (b - a).$$

Consider g which is obtained by subtracting the line $\ell(x)$ through $(a, f(a))$ and $(b, f(b))$ from the function f i.e., $g(x) = f(x) - \ell(x)$.



Exercise 79

Show that g satisfies Rolle's theorem and conclude with the mean value theorem.

Variation

We now make the link between global variation and derivative.

Definition 15

Let f be a real function defined on an interval I.

*(1) The function f is **increasing on** I if $f(x) \leq f(y)$, whenever $x < y$ in I.*

*(2) The function f is **decreasing on** I if $f(x) \geq f(y)$, whenever $x < y$ in I.*

If the inequalities are strict, then we say that the function is strictly increasing or strictly decreasing.

Theorem 24 (Variation and Derivative)

Let f be a real function differentiable on an interval I. Then

(1) If $f'(x) \geq 0$ (> 0) whenever $x \in I$ then f is (resp. strictly) increasing on I.

(2) If $f'(x) \leq 0$ (< 0) whenever $x \in I$ then f is (resp. strictly) decreasing on I.

(3) If $f'(x) = 0$ whenever $x \in I$ then f is constant on I.

The converse is obvious: if f is increasing at a, then $f'(a) > 0$.

Exercise 80

Prove theorem 24 using the mean value theorem.

Exercise 81

Prove the following theorem:

Theorem 25 (Uniqueness, up to an Additive Constant)
Let f and g be functions and I an interval.

$$f' = g' \iff \textit{there is a real number } C \textit{ such that } f = g + C$$

Theorem 27 is one direction of theorem 25. You will need theorem 24 (page 59) for the other direction.

The differential

It can be convenient to write dx instead of Δx, with the understanding that $dx \neq 0$.

Definition 16
*Let f be a real function differentiable on an interval around a. Let dx be ultrasmall. The **differential of f at a**, written $df(a)$, is*

$$df(a) = f'(a) \cdot dx.$$

Thus

$$\frac{df(a)}{dx} = f'(a)$$

or still (if we use $y = f(a)$)

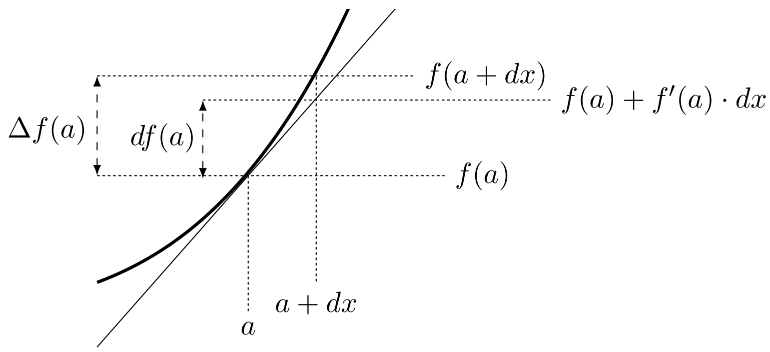
$$\frac{dy}{dx} = y'$$

.

If f is differentiable the following holds:

$$\frac{\Delta f(a)}{dx} \simeq \frac{df(a)}{dx}$$

Whereas $\Delta f(a)$ is the variation of the function, the differential is the variation along the tangent line.



The chain rule can be written, for y as function of x and z as function of y:

$$dz = z' \cdot dy = z' \cdot y' \cdot dx$$

hence

$$\frac{dz}{dx} = z' \cdot y'$$

9
Integrals

9.1 Area under a positive curve

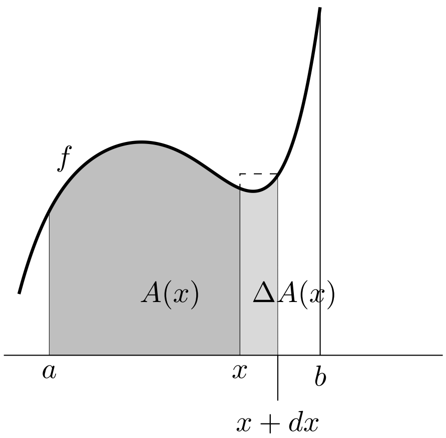
Theorem 26
Let f be a non-negative function continuous on $[a;b]$ and smooth on $]a,b[$. Then the function

$$A : x \mapsto A(x),$$

where $A(x)$ is the area under the curve of f between a and x has the following properties

(1) $A'(x) = f(x)$*, whenever* $x \in [a;b]$*.*

(2) $A(a) = 0$*.*



Exercise 82
Prove theorem 26.
Hint: Context is a, b, f and x. Let dx be ultrasmall. As f is smooth on $[x; x+dx]$ the function f reaches its maximum, say $(x_M, f(x_M))$, and its minimum, say $(x_m, f(x_m))$ in that interval (theorem 20).
Show that $\Delta A(x)$ is bounded above and below, that these bounds are ultraclose, then conclude.

For $dx > 0$. On $[x, x+dx]$ the function reaches a max and a min. Hence the slice $\Delta A(x)$ is bounded below by the rectangle $f(x_m) \cdot dx$ and above by the rectangle $f(x_M) \cdot dx$, hence

$$f(x_m) \cdot dx \leq \Delta A(x) \leq f(x_M) \cdot dx$$

then, since x_m and x_M are in $[x, x+dx]$, dividing by dx we get:

$$f(x) \simeq f(x_m) \leq \frac{\Delta A(x)}{dx} \leq f(x_M) \simeq f(x) \Rightarrow \frac{\Delta A(x)}{dx} \simeq f(x)$$

By taking $dx < 0$ we notice that the area decreases and the the inequalites are reversed, hence, not depending on dx we have

$$\frac{\Delta A(x)}{dx} \simeq f(x) \Rightarrow A'(x) = f(x)$$

$A(a) = 0$ by the definition that it is the area between a and a.

Notation

$$A(b) - A(a) \text{ is written } A(x)\Big|_a^b$$

9.2 Antiderivative

Definition 17 (Antiderivative)
An antiderivative of a function f is a function A such that $A'(x) = f(x)$.

Newton assumed that gravitation is a constant acceleration. Given such an acceleration, how can one find the equation of position with respect to time?

What is the area under a curve and what is the relation between measuring areas and retrieving the function of position when velocity is known?

Exercise 83
Prove the following theorem:

Theorem 27
If A is an antiderivative of f, then for any constant $C \in \mathbb{R}$, $A + C$ is also an antiderivative of f.

Exercise 84
Find the antiderivatives for the following:

(1) $x \mapsto 3x$

(2) $x \mapsto x^2$

(3) $x \mapsto 5$

(4) $t \mapsto 3t + 5$

(5) $u \mapsto u^2 + 3u + 5$

(6) $v \mapsto v^3$

(7) $x \mapsto \frac{1}{\sqrt{x}}$

Check your results by differentiating them.

Exercise 85

Using $A' = f$ and $A(a) = 0$:

(1) Calculate the area between the curve and the x-axis for $y = x^2$ from $x = -5$ to $x = 5$.

(2) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = 0$ to $x = 3$.

(3) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = -2$ to $x = 0$.

(4) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = -10$ to $x = 10$.

Exercise 86

Calculate the area between $y = 5x^4 - 3x^3 + 2x^2 - 10$ and the x-axis from $x = -1$ to $x = 1$.

9.3 A ∫um of ∫lices

Exercise 87

Let $g : x \mapsto x^2$, $a = 0$ and $b = 5$.

(1) Cut the interval $[a; b]$ into an ultralarge number N of pieces. Put all these pieces together again – add all their lengths. What is the result?

Write this using the symbol for a sum i.e., sum for $k = 0$ to $N - 1$.

(2) For each $\Delta x = \frac{b-a}{N}$ there is a corresponding Δy. Add all the Δy between $f(a)$ and $f(b)$. Find the result.

(3) Use the microscope equation to express Δy in terms of y or y'. Add all these terms. Find the result.

The (vertical) variation of f between a and b is written $f(x)\Big|_a^b$

Let N be an ultralarge natural number and let $dx = \frac{5-0}{N} = \frac{5}{N}$. Set $x_k = k \cdot dx$. Each pice goes from x_k to x_{k+1} for some k. The length is dx.

$$\sum_{k=0}^{N-1} dx = N \cdot dx = N \cdot \frac{5}{N} = 5$$

We have the telescoping sum:

$$f(5) - f(0) = \sum_{k=0}^{N-1} f(x_{k+1}) - f(x_k) = \sum_{k=0}^{N-1} \Delta f(x_k) \tag{*}$$

$$= \sum_{k=0}^{N-1} (f'(x_k) \cdot dx + \varepsilon_k \cdot dx)$$

$$= \sum_{k=0}^{N-1} f'(x_k) \cdot dx + \sum_{k=0}^{N-1} \varepsilon_k \cdot dx$$

For the second sum, let $\varepsilon = \max\{\varepsilon_k\}$ then

$$\sum_{k=0}^{N-1} \varepsilon_k \cdot dx \leq \sum_{k=0}^{N-1} \varepsilon \cdot dx = \varepsilon \cdot \sum_{k=0}^{N-1} dx = 5 \cdot dx \simeq 0$$

Hence we get the relation:

$$x^2\Big|_0^5 \simeq \sum_{k=0}^{N-1} f'(x_k) \cdot dx = \sum_{k=0}^{N-1} 2 \cdot x_k \cdot dx$$

For the area under x^2 between $x = 0$ and $x = 5$, we look at a sum of slices of area. This will give the total variation of the area.

$$A = \sum_{k=0}^{N-1} \Delta A(x_k)$$

This equation is the same as (*) above. Assuming $A' = f$ as shown in theorem 26, we have

$$A(x)\Big|_b^a \simeq \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

⚠ Questions: How can we be sure that the function A exists and how do we define the area under a function?

We will now in fact reverse the process: define these sums and then define the area using these.

Definition 18
*Let f be a real function defined on $[a;b]$. Let n be a positive integer. Let $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$, for $i = 0, \dots, n$. We say that f **is integrable on** $[a;b]$ if there is an observable I such that for any ultralarge integer n with $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$, for $i = 0, \dots, n$, we have*

$$\sum_{i=0}^{n-1} f(x_i) \cdot dx \simeq I.$$

If such an I exists, it is called the integral of f between a and b; written

$$\int_a^b f(x) \cdot dx.$$

Note that this sum is defined whether f is positive or not.

preliminary results

Exercise 88
Prove the following preliminary results

Lemma 1
Let $dx = \frac{b-a}{N}$ for ultralarge N, and all $\varepsilon_i \simeq 0$. Then

$$\sum_{i=0}^{N-1} \varepsilon_i \cdot dx \simeq 0$$

Let $\varepsilon = \max\{\varepsilon_i \mid 0 \leq i \leq N-1\}$

$$\sum_{i=0}^{N-1} \varepsilon_i \cdot dx \leq \sum_{i=0}^{N-1} \varepsilon \cdot dx = \varepsilon \cdot \sum_{i=0}^{N-1} dx = N \cdot dx = N \cdot \frac{b-a}{N} = \underbrace{\varepsilon}_{\simeq 0} \cdot \underbrace{(b-a)}_{\text{observable}} \simeq 0$$

Lemma 2
Let f be an function continuous on $[a;b]$. Let $\frac{1}{N} \simeq 0$, $dx = \frac{b-a}{N}$ and $x_k = a + k \cdot dx$, then there exists a point $c \in [a;b]$ such that

$$f(c) \cdot (b-a) = \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

For f continuous on $[a,b]$, f has a maximum $f(x_M)$ and a minimum $f(x_m)$.

$$\sum_{k=0}^{N-1} f(x_m)\cdot dx \leq \sum_{k=0}^{N-1} f(x_k)\cdot dx \leq \sum_{k=0}^{N-1} f(x_M)\cdot dx$$

hence

$$f(x_m)\cdot \sum_{k=0}^{N-1} dx \leq \sum_{k=0}^{N-1} f(x_k)\cdot dx \leq f(x_M)\cdot \sum_{k=0}^{N-1} dx$$

then

$$f(x_m)\cdot (b-a) \leq \sum_{k=0}^{N-1} f(x_k)\cdot dx \leq f(x_M)\cdot (b-a)$$

or

$$f(x_m) \leq \frac{\sum_{k=0}^{N-1} f(x_k)\cdot dx}{b-a} \leq f(x_M)$$

Since f is assumed continuous on $[a,b]$, by the intermediate value theorem, it reaches all intermediate values, hence there is a $c \in [a,b]$ such that

$$f(c) = \frac{\sum_{k=0}^{N-1} f(x_k)\cdot dx}{b-a}$$

Theorem 28
If f is continuous an $[a;b]$ then f is integrable on $[a;b]$

No proof of this in maths 1... see the maths 2 handout for why. It is quite simple to show that for any ultralarge N , the sum has an observable neighbour (previous theorem). It is technically more of a mess to show that it does not depend on the choice of the ultralarge N .

Fundamental Theorem of Calculus

Definition 19
Let f be a real function defined on $[a;b]$. Let n be a positive integer. Let $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i \cdot \Delta x$, for $i = 0, \ldots, n$. We say that f ***is integrable on*** *$[a;b]$ if there is an observable I such that for any ultralarge integer n with $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i \cdot \Delta x$, for $i = 0, \ldots, n$, we have*

$$\sum_{i=0}^{n-1} f(x_i) \cdot \Delta x \simeq I.$$

If such an I exists, it is called the integral of f between a and b; written

$$\int_a^b f(x) \cdot dx.$$

Note that this sum is defined whether f is positive or not.

Note also that the integral is written with dx and not Δx.

Theorem 29 (Additivity of the integral)
Let f be a real function continuous on $[a;c]$ and $b \in [a;c]$. Then

$$\int_a^b f(x) \cdot dx + \int_b^c f(x) \cdot dx = \int_a^c f(x) \cdot dx.$$

> No proof: there is a hidden difficulty here: Using even divisions of the interval makes most of the proof easier, but for additivity we would need continuity of the integral. See the maths 2 handout.

Theorem 30 (Fundamental theorem of Calculus (part 1))
If f is a continuous function on $[a,b]$ then

$$F(x) = \int_a^x f(t) \cdot dt$$

is an antiderivative of f on $]a,b[$ and the only one satisfying $F(a) = 0$.

Exercise 89

Prove theorem 30 starting with the definition of the derivative applied to the integral. By theorem 28, it is integrable.

By additivity:

$$F(x+dx) - F(x) = \int_x^{x+dx} f(t)\cdot dt$$

By lemma 2, there is a $c \in [x, x+dx]$ such that

$$\int_x^{x+dx} f(t)\cdot dt = f(c)\cdot dx$$

hence

$$\frac{\Delta F(x)}{dx} = f(c) \simeq f(x)$$

by continuity of f since $x \simeq c$. hence $F'(x) = f(x)$.

Theorem 31 (Fundamental theorem of Calculus (part 2))
Let f be a function continuous on $[a;b]$. Let F be an antiderivative of f on $[a;b]$. Then

$$\int_a^b f(x)\cdot dx = F(b) - F(a).$$

Notation: we write

$$F(x)\Big|_a^b = F(b) - F(a).$$

The $\int$ symbol is an elongated S and stands for the latin word "summa": a sum, since it can also be shown that instead of finding the area as a variation, it is a ∫um of ∫lices.

The method used in the proof can also be seen as looking at the link between the global variation of a function F and its derivative f.

Exercise 90

Consider the variation of F between a and b.
Let $n \in \mathbb{N}$ such that $1/N \simeq 0$ and $dx = \frac{b-a}{N}$ and $x_k = a + k\cdot dx$.
Then clearly, we have

$$F(b) - F(a) = \sum_{k=0}^{N-1} \Delta F(x_k)$$

Here the context is f, a, b – not necessarily any given x_j!

(1) On each interval $[x_k, x_{k+1}]$ (which is also in the form $[x_k, x_k + dx]$) there is a c such that

$$F(x_k + dx) - F(x_k) = f(c)\cdot dx,$$

Why is this? By what theorem?

(2) Explain why we have $f(c) \simeq f(x_k)$.

(3) Conclude by explaining why:

$$\sum_{k=0}^{N-1} F(x_k+dx) - F(x_k) = \sum_{k=0}^{N-1} f(x_k)\cdot dx + \sum_{k=0}^{N-1} \varepsilon_k \cdot dx \simeq \sum_{k=0}^{N-1} f(x_k)\cdot dx$$

Hence, the global variation of F between a and b is, up to an ultrasmall value, the sum of $F'(x_i) \cdot dx$ provided F' is continuous on $[a, b]$.

If bounds are given, the integral represents a value: it is a **definite integral**. If no bounds are given, it represents an antiderivative: it is an **indefinite integral**.

Exercise 91
Let $f : x \mapsto x^2$,
Calculate the area under $f'(x)$ between $x = 0$ and $x = 5$.

Exercise 92
Show that for a definite integral, it does not matter which antiderivative is chosen.

Exercise 93
What conditions would a function need to satisfy in order to be non-integrable? Give such a function.

Exercise 94
A constant function $f : x \mapsto C$ from a to b defines a rectangle. Check that the area under f is the "usual" formula: $(b - a) \cdot C$

Exercise 95
The function $y = x$ defines a triangle. Show that the area of the triangle from 0 to a yields the "usual" result for the area of a triangle.

Exercise 96
Sketch the curve of $f : x \mapsto x^2$ and $g : x \mapsto x^3$. Calculate the points where $f(x) = g(x)$
Calculate the geometric area of the closed surface between the two curves.

Integration Rules

Theorem 32 (Linearity of the integral)
Let f and g be real functions integrable on $[a; b]$. Let λ be a real number. Then

(1)

$$\int_a^b (\lambda \cdot f(x)) \cdot dx = \lambda \cdot \int_a^b f(x) \cdot dx$$

(2)

$$\int_a^b (f(x) + g(x)) \cdot dx = \int_a^b f(x) \cdot dx + \int_a^b g(x) \cdot dx.$$

Note that if f and g are integrable then all linear combinations of f and g are integrable.

Exercise 97

Prove theorem 32.

Exercise 98

For each of the following functions, find the general form of the antiderivative:

(1) $f : x \mapsto 8\sqrt{x}$

(2) $f : t \mapsto 3t^2 + 1$

(3) $f : t \mapsto 4 - 3t^3$

(4) $f : s \mapsto 7s^{-3}$

(5) $f : x \mapsto (x - 6)^2$

(6) $f : y \mapsto y^{\frac{3}{2}}$

(7) $f : x \mapsto |x|$

(8) $f : u \mapsto u^2 + u^{-2}$

(9) $f : x \mapsto 4$

(10) $f : t \mapsto t$

(11) $f : z \mapsto \dfrac{2}{z^2}$

Check your results by differentiating them.

Exercise 99

(1) If $F'(x) = x + x^2$ for all x, find $F(1) - F(-1)$.

(2) If $F'(x) = x^4$ for all x, find $F(2) - F(1)$.

(3) If $F'(t) = t^{\frac{1}{3}}$ for all t, find $F(8) - F(10)$.

Exercise 100

The following computation may seem correct: $\int_{-1}^{1} x^{-2}dx = -\frac{1}{x}\Big|_{-1}^{1} = -2$ yet there is no $x \in [-1, 1]$ such that $f(x) < 0$. Why is this not so?

Exercise 101

In the following problems an object moves along the y axis. Its velocity varies with respect to the time. Find how far the object moves between the given times t_0 and t_1.

(1) $v = 2t + 5$ $t_0 = 0$ $t_1 = 2$

(2) $v = 4 - t$ $t_0 = 1$ $t_1 = 4$

(3) $v = 3$ $t_0 = 2$ $t_1 = 6$

(4) $v = 3t^2$ $t_0 = 1$ $t_1 = 3$

(5) $v = 10t^{-2}$ $t_0 = 1$ $t_1 = 100$

Theorem 33 (Integration with inside derivative)
Let f and g be real functions differentiable on $[a; b]$ such that f' and g' are continuous on $[a; b]$. Then

$$\int_a^b f'(g(x)) \cdot g'(x) \cdot dx = f(g(x))\Big|_a^b.$$

Exercise 102
Prove theorem 33.

Variable substitution

In this section, the differential notation and the chain rule are used extensively.

Consider $\int_a^b f(g(x)) \cdot dx$.

If we write $g(x) = u$ written then $\frac{du}{dx} = g'(x)$ and $dx = \frac{du}{u'}$,

$f(x) \cdot dx$ becomes $\frac{f(u)}{u'} \cdot du$ and the bounds must be changed to a_1 and b_1 so that $a_1 = g(a)$ and $b_1 = g(b)$

Example: For

$$\int_1^2 2x \cdot (x^2+1)^2 \cdot dx$$

(Considering that $2x$ is the inside derivative, the antiderivative can be seen to be $\frac{(x^2+1)^3}{3}$. Here we consider another approach by variable substitution.)

Let $u = x^2 + 1$, then $\frac{du}{dx} = 2x$ hence $dx = \frac{du}{2x}$.

Then

$$2x \cdot (x^2+1)dx = 2x \cdot u^2 \cdot \frac{du}{2x} = u^2 \cdot du$$

As for the bounds: if $x = 1$ then $u = x^2 + 1 = 2$ and if $x = 2$ then $u = 4 + 1 = 5$, hence

$$\int_1^2 2x \cdot (x^2+1)^2 \cdot dx = \int_2^5 u^2 \cdot du = \frac{u^3}{3}\Big|_2^5$$

which gives $(125 - 8)/3 = 39$

Compare with $\frac{(x^2+1)^3}{3}\Big|_1^2 = (5^3 - 2^3)/3 = 39$

> Since dx is a quantity and $du = u'dx$ also, this theorme can be avoided as such and variable by substitution can be given as a step by step method.

Example: Let

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx.$$

Consider the variable change $u = 1 + \sqrt{x}$. Then $x = (u-1)^2 = g(u)$, the derivative of g is continuous. If $x = 0$ then $u = 1$ and if $x = 1$ then $u = 2$. Moreover $f(g(u)) = \sqrt{u}$ and

$$dx = 2 \cdot (u-1) \cdot du.$$

Replacing all terms we obtain

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx = 2\int_1^2 \sqrt{u} \cdot (u-1) \cdot du = 2\int_1^2 \left(u^{3/2} - u^{1/2}\right) \cdot du$$

so that

$$2\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Bigg|_1^2=\frac{8+8\sqrt{2}}{15}.$$

As g has an inverse which is $x\mapsto 1+\sqrt{x}$ and is differentiable (except at $x=0$), we can revert to the variable x and find an antiderivative:

$$\int\sqrt{1+\sqrt{x}}\cdot dx=\frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5-\frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3+C.$$

Exercise 103

Calculate

$$\int_0^1\sqrt{5x+2}\cdot dx.$$

Use $u=5x+2$. Calculate du, change the bounds, calculate the integral.

Same integral. Use $v=\sqrt{5x+2}$

The difficulty is usually to find which variable substitution is best.

Exercise 104

Use variable substitution to evaluate the following:

(1) $\displaystyle\int_0^{10} \frac{1}{(2x+2)^2} \cdot dx$

(2) $\displaystyle\int (3-4z)^6 \cdot dz$

(3) $\displaystyle\int_{-1}^{1} 2t\sqrt{1-t^2} \cdot dt$

(4) $\displaystyle\int_a^b \sqrt{3y+1} \cdot dy$

(5) $\displaystyle\int \frac{4y}{(2+3y^2)^2} \cdot dy$

(6) $\displaystyle\int_{-2}^{2} x(4-5x^2)^2 \cdot dx$

(7) $\displaystyle\int (1-x)^{\frac{3}{2}} \cdot dx$

Practice exercise 13 Answer page 88

(1) $\displaystyle\int_0^1 \frac{u}{\sqrt{1-u^2}} \cdot du$

(2) $\displaystyle\int_1^2 \frac{u}{\sqrt{1-u^2}} \cdot du$

(3) $\displaystyle\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx$

(4) $\displaystyle\int_0^{10} t(t^2+3)^{-2} \cdot dt$

(5) $\displaystyle\int_{\sqrt{6}}^{5} x(x^2+2)^{\frac{1}{3}} \cdot dx$

(6) $\displaystyle\int_{-1}^{1} \frac{x^2}{(4-x^3)^2} \cdot dx$

(7) $\displaystyle\int_1^2 \frac{1}{t^2\sqrt{1+\frac{1}{t}}} \cdot dt$

Antiderivative of $x \mapsto \dfrac{1}{x}$

Let n be a positive integer. From $(x^{n+1})' = (n+1) \cdot x^n$ we can deduce

$$\int x^n \cdot dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1.$$

Hence an antiderivative of $x \mapsto \dfrac{1}{x}$ is not a particular case of this formula.

Exercise 105

Let f be an antiderivative of $x \mapsto \frac{1}{x}$ (why is there one?) Then f is strictly increasing (why?) and so it has an inverse, call it g. Show that this implies $g'(x) = g(x)$.

Exercise 106

Let $a, b > 0$. Use the substitution $u = \frac{t}{a}$ to show that (considering f to be the antiderivative of $\frac{1}{x}$.)

$$\int_a^{a \cdot b} \frac{1}{t} \cdot dt = \int_1^b \frac{1}{u} \cdot du.$$

Deduce that $f(a \cdot b) = f(a) + f(b)$.

Exercise 107

Let $a > 0$ and b a rational number. Show that (considering f to be the antiderivative of $\frac{1}{x}$.)

$$f(a^b) = b \cdot f(a).$$

(To find the substition, consider the transformation of the bounds.)

Exercise 108

What kind of function has the properties $f(a \cdot b) = f(a) + f(b)$ and $f(a^b) = b \cdot f(a)$?

Theorem 34

The antiderivative f of $\frac{1}{x}$ satisfies the following properties

- $x \simeq 0_+ \Rightarrow f(x)$ *is ultralarge and negative*
- x *is ultralarge positive* $\Rightarrow f(x)$ *is ultralarge positive.*

Exercise 109

Prove theorem 34. Hint: for ultralarge x use ultralarge N such that $2^N \leq x$.

Definition 20

The ***natural logarithm*** *is the function* $\ln :]0; +\infty[\to \mathbb{R}$ *defined by*

$$x \mapsto \int_1^x \frac{1}{t} \cdot dt.$$

Definition 21

We define e to be the unique number such that

$$\ln(e) = 1.$$

e is an irrational number whose first digits are

$$e = 2.71828\ldots$$

Definition 22

The ***exponential*** *function* $\exp : \mathbb{R} \longrightarrow]0; +\infty[$ *is defined as the inverse of* $\ln$.

We have, for rational x, that $a^x = \exp(x\ln(a))$, hence $e^x = \exp(x)$. For irrational x, we **define** a^x to be $\exp(x\ln(a))$ hence also $e^x = \exp(x)$ for all x.

We also have $\ln(a^y) = y \cdot \ln(a)$ for all y. Writing $x = a^y$ we get $\ln(x) = \log_a(x) \cdot \ln(a)$ so $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

The following property makes it a remarkable function.

Theorem 35

$$(\exp(x))' = \exp(x).$$

(this was proven by exercise 105).

Exercise 110

Let f be a positive real function whose derivative is continuous. Calculate:

$$\int \frac{f'(x)}{f(x)} \cdot dx$$

Exercise 111

Let f be a positive real function whose derivative is continuous. Calculate:

$$\int f'(x) \cdot e^{f(x)} \cdot dx$$

Exercise 112

(1) Differentiate $\ln(x)$.

(2) Differentiate e^x.

(3) Integrate $x \mapsto e^x$.

(4) Differentiate the function $x \mapsto \ln(\ln(x))$.

(5) Differentiate the function $x \mapsto \ln(x^a)$ (Note that a is not the variable!)

(6) Differentiate the function $x \mapsto \ln(a^x)$.

(7) Differentiate $x \mapsto e^{x^2}$.

(8) Using the fact that $u = e^{ln(u)}$ (if $u > 0$) differentiate $x \mapsto a^x$ (for $a > 0$ and $x > 0$).

(9) Same idea: Differentiate the function $x \mapsto x^x$.

Exercise 113

Differentiate $\ln(|x|)$.

This proves the following extension:

Theorem 36

The antiderivative of $\frac{1}{x}$ is $\ln(|x|) + K$ for some constant K.

Practice exercise 14 Answer page 88

Find the antiderivatives of the following functions:

- $f_a : x \mapsto 5x^4 - 2x + 4$
- $f_b : x \mapsto x^3 - 5x^2 + 3x - 2$
- $f_c : x \mapsto 2x - 1$
- $f_d : x \mapsto \frac{5}{4}x^4 - \frac{3}{4}x^2 + \frac{5}{2}x + \frac{3}{2}$
- $f_e : x \mapsto 2x + 1 - \frac{1}{x^2}$
- $f_f : x \mapsto 3 + \frac{2}{x^2} - \frac{5}{x^3}$
- $f_g : x \mapsto x^3 + \frac{1}{x^2}$
- $f_h : x \mapsto \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
- $f_i : x \mapsto \frac{1}{\sqrt{x}} + \sqrt{x}$
- $f_j : x \mapsto (x+1)^2$
- $f_k : x \mapsto 15(3x-2)^4$
- $f_l : x \mapsto (2x+1)^3$
- $f_m : x \mapsto (3-x)^{11}$
- $f_n : x \mapsto (3-4x)^4$
- $f_o : x \mapsto \sqrt{3x-2}$
- $f_p : x \mapsto \frac{1}{\sqrt{x-1}}$
- $f_q : x \mapsto 4x(3-x^2)^5$
- $f_r : x \mapsto (2x-3)(x^2-3x+1)^4$
- $f_s : x \mapsto (3x^2-4x+1)(x^3-2x^2+x+3)^2$
- $f_t : x \mapsto (4x^2-5x)^2(16x-10)$
- $f_u : x \mapsto (3x-1)(3x^2-2x+5)^3$
- $f_v : x \mapsto \frac{2x}{(x^2+1)^2}$
- $f_w : x \mapsto \frac{2x+1}{(x^2+x+3)^2}$
- $f_x : x \mapsto x\sqrt{x^2+1}$
- $f_y : x \mapsto \frac{3x^2}{\sqrt{9+x^3}}$
- $f_z : x \mapsto (3x^2+1)\sqrt{x^3+x+2}$
- $f_A : x \mapsto e^{2x}$
- $f_B : x \mapsto \frac{1}{e^{3x}}$
- $f_C : x \mapsto xe^{-x^2}$
- $f_D : x \mapsto 2^{-x}$
- $f_E : x \mapsto e^{2x}\sqrt{1+e^{2x}}$
- $f_F : x \mapsto x^2e^x$
- $f_I : x \mapsto \frac{1}{2x+3}$
- $f_J : x \mapsto \frac{2x}{x-1}$
- $f_K : x \mapsto \frac{x-1}{x+1}$
- $f_L : x \mapsto (\ln(x))^2$
- $f_N : x \mapsto \ln(x)$
- $f_O : x \mapsto \frac{x}{x+1}$
- $f_P : x \mapsto \frac{1}{x\ln(x)}$

Applications of the Integral

Mean value of a function

The mean value is unambiguous when we consider n points, where n is a positive integer. We now show that defining the mean value of a continuous function on $[a; b]$ as

$$\frac{1}{b-a}\int_a^b f(x)\cdot dx$$

is a natural extension of this concept.

Consider a continuous function f and the interval $[a; b]$. Context is a, b and f. Let N be a positive ultralarge integer. Let $dx = (b-a)/N$ and $x_i = a + i \cdot dx$, for $i = 1, \ldots, N$. Then the mean value of the function can be approximated by the mean value of the N points $f(x_i)$, $i = 0, \ldots, N-1$. But

$$\frac{\sum_{i=0}^{N-1} f(x_i)}{N} = \frac{dx}{b-a}\sum_{i=0}^{N-1} f(x_i) = \frac{1}{b-a}\sum_{i=0}^{N-1} f(x_i)\cdot dx \simeq \frac{1}{b-a}\int_a^b f(x)\cdot dx,$$

since f is continuous on $[a; b]$.

The mean is the part of this number which is observable i.e., the integral. We therefore define:

Definition 23
*The **mean value** of a function f continuous on $[a; b]$ is*

$$\frac{1}{b-a}\int_a^b f(x)\cdot dx.$$

The mean value is a number μ such that the area under the curve is equal to $\mu \cdot (b-a)$, i.e., the height of a rectangle of basis $(b-a)$ whose (oriented) area is equal to the integral.

Theorem 37
If f is a function continuous on $[a; b]$, then there exists a point $c \in [a; b]$ such that $f(c)$ is the mean value of the function on $[a; b]$.

Note that theorem 37 is a restatement of the mean value theorem, for the antiderivative of f. When we claim that there is a $c \in [a; b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\cdot dx,$$

we are in fact asserting that there is a $c \in [a; b]$ such that

$$f(c)\cdot(b-a) = \int_a^b f(x)\cdot dx = F(b) - F(a),$$

and as $F'(x) = f(x)$, we conclude that there is a $c \in [a; b]$ such that $F'(c)\cdot(b-a) = F(b) - F(a)$.

Exercise 114

Calculate the mean value of $x \mapsto x^2$ on $[-4; 4]$.

Exercise 115

Calculate the mean value of $x \mapsto x^3$ on $[-4; 4]$.

Exercise 116

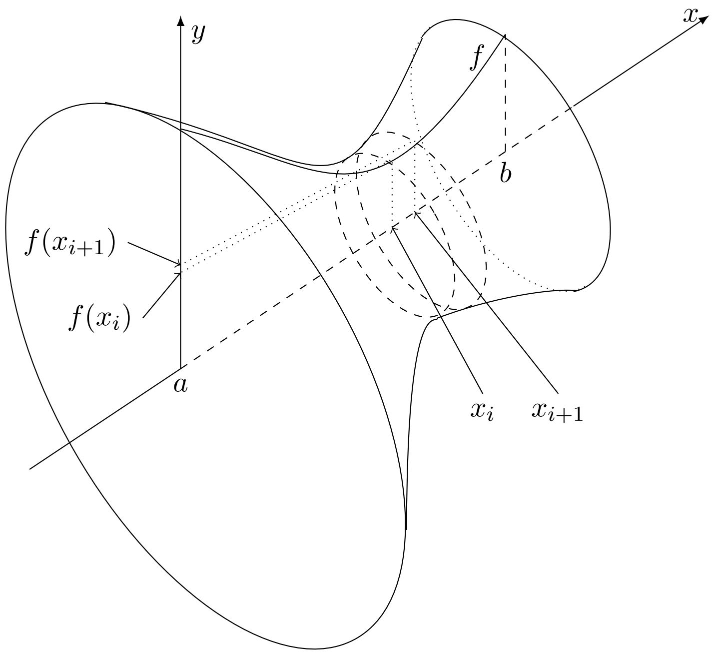
Let $f : x \mapsto x^2$ and the interval $[0; t]$. Find the value of t such that the mean value of f over the interval is equal to π.

Exercise 117

An object falling on earth satisfies the equation $d(t) = \frac{1}{2}gt^2$ where $g \approx 9.81[m/s^2]$, t is the time in seconds and $d(t)$ is the vertical distance.

If an object falls for $10s$, what is its average distance from its initial point?

Solid of Revolution



Exercise 118

An area is calculated by approximating the surface by ultrasmall rectangles. To find the formula for the volume of a solid of revolution, proceed in the same manner: consider that the solid is ultraclose to an ultralarge number of ultrathin disks. Find the formula for the volume of a solid of revolution given by a function f.

Exercise 119

Evaluate the volume of the solid of revolution of $y = \dfrac{1}{x}$ around the x-axis between $x = 1$ and $x = 10$.

Arc length

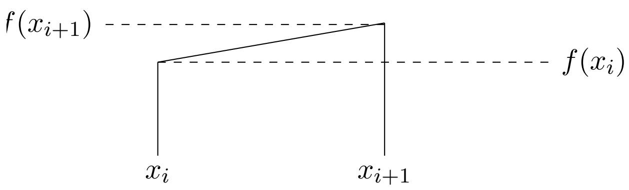
Exercise 120

Approximating the length of a curve by ultrasmall straight lines leads to the following definition. Explain why it is a reasonable definition (using the drawing).

Definition 24

Let $f : [a; b] \to \mathbb{R}$ be smooth. Then the graph of f has length

$$L = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx.$$



Exercise 121

Find the lengths of the following curves:

(1) $y = 2x^{3/2} \quad 0 \leq x \leq 1$

(2) $y = \dfrac{2}{3}(x+2)^{\frac{3}{2}} \quad 0 \leq x \leq 3$

---

10 Limits

If we want to study the behaviour of f in the neighbourhood of a, the function f must be defined *around* a, but not necessarily at a. If the function is defined in a neighbourhood of a, by closure, it is possible to use a neighbourhood defined by observable bounds. Hence $f(x)$ must exist for $x \simeq a$ but $f(a)$ does not necessarily exist. Context is f and a.

Definition 25
*A **deleted interval of** a is an interval around a not containing a.*

The limit of f at a is the value that f should take in order to be continuous at a.

Definition 26
*Let f be a real function defined on a deleted interval of a. Context is f and a. We say that f **has a limit at** a if there exists an observable number L such that if we had $f(a) = L$ then f would be continuous at a,*

In other terms, if there is an observable number L such that

$$x \simeq a \Longrightarrow f(x) \simeq L.$$

Of course, by this definition, if f *is* continuous at a, then the limit of f at a is $f(a)$.
The definition of limit can also be interpreted in the following way:

If f has a limit at a then it is the observable neighbour of $f(a + dx)$.
If L is the limit of f at a we write

$$f(a + dx) \simeq L$$

or

$$\lim_{x \to a} f(x) = L,$$

Exercise 122
Calculate

$$\lim_{x \to 3} \frac{2x^2 - 7x + 3}{x - 3}.$$

Show that it is equal to

$$\lim_{h \to 0} \frac{2(3 + h)^2 - 7(3 + h) + 3}{(3 + h) - 3}.$$

Exercise 123

Consider the signum function sgn, defined by

$$\text{sgn} : x \mapsto \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

Check that sgn is defined around 0. Does it have a limit at 0?

One Sided Limits

A function is defined **on the left** (respectively **on the right**) of a, if $f(x)$ exists for $x \simeq a$, $x < a$ (respectively $x \simeq a$, $x > a$).

Definition 27

*Let f be a real function defined on the left of a. The function f **has a limit on the left of** a if there is an obervable number L such that*

$$x \simeq a \textit{ and } x < a \implies f(x) \simeq L.$$

If the limit on the left exists it is unique (it is the observable neighbour of $f(x)$). We write:

$$\lim_{x \to a_-} f(x) = L, \quad \text{or} \quad x \simeq a_- \Rightarrow f(x) = L.$$

The symbol a_- indicates that we choose numbers less than a.
Similarly we define the **limit on the right of** a and write:

$$\lim_{x \to a_+} f(x) = L, \quad \text{or} \quad x \simeq a_+ \Rightarrow f(x) = L.$$

The symbol a_+ indicates that we choose numbers greater than a.

The limit is only a rewriting. The "equal" sign used is there to say that the limit is the value that the function can be ultraclose to. When a limit appears in a problem, the first thing to do is to rewrite it in terms of ultracloseness.

The symbol "∞" can be used to indicate that the function takes ultralarge values. Since if a function has a maximum, by closure, the maximum would be observable, the fact that it reaches ultralarge values implies that it has no maximum, hence that the intervakl of possible results is infinite.

Practice exercise 15 Answer page 88

Calculate the following limits. The answer should be a number, $+\infty$, $-\infty$ or "does not exist"

(1) $\lim\limits_{x\to\infty} \dfrac{6x-4}{2x+5}$

(2) $\lim\limits_{x\to\infty} x^3-10x^2-6x-2$

(3) $\lim\limits_{x\to\infty} \dfrac{x^2-x+4}{3x^2+2x-3}$

(4) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+2}}{\sqrt{3x+1}}$

(5) $\lim\limits_{x\to\infty} x-\sqrt{x}$

(6) $\lim\limits_{x\to\infty} \sqrt[3]{x+2}$

(7) $\lim\limits_{x\to 0_-} 1+\dfrac{1}{x}$

(8) $\lim\limits_{x\to 0} \dfrac{1}{x^2}-\dfrac{1}{x}$

(9) $\lim\limits_{x\to 0} \dfrac{1+2x^{-1}}{7+x^{-1}-5x^{-2}}$

(10) $\lim\limits_{x\to 2} \dfrac{1-x}{2-x}$

(11) $\lim\limits_{x\to 3_+} \dfrac{x+1}{(x-2)(x-3)}$

(12) $\lim\limits_{x\to 3} \dfrac{x+1}{(x-2)(x-3)}$

(13) $\lim\limits_{x\to 1} \dfrac{3x^2+4}{x^2+x-2}$

(14) $\lim\limits_{x\to 2_+} \dfrac{x^2+4}{x^2-4}$

(15) $\lim\limits_{x\to\infty} \sqrt{x^2+1}-x$

(16) $\lim\limits_{x\to-\infty} \sqrt{x^2+1}-x$

(17) $\lim\limits_{x\to\infty} \sqrt{x^2-3x+2}-\sqrt{x^2+1}$

(18) $\lim\limits_{x\to\infty} \sqrt[3]{x+4}-\sqrt[3]{x}$

11
Curve Sketching

Now the rules of de l'Hospital may be also required. The functions may include any combination of functions studied up to now. Some functions may be difficult.

Sketch the curves of the following.

Practice exercise 16 Answer page 89

- $g_1 : x \mapsto x\ln(x)$
- $g_2 : x \mapsto \dfrac{x}{\ln(x)}$
- $g_3 : x \mapsto \dfrac{e^x}{\ln(x)}$
- $g_4 : x \mapsto \dfrac{e^x}{1+e^x}$
- $g_5 : x \mapsto \dfrac{1}{1+e^x}$
- $g_6 : x \mapsto \ln(x^2+1)$
- $g_7 : x \mapsto \dfrac{e^x}{x-2}$
- $g_8 : x \mapsto e^{-x^2}$
- $g_9 : x \mapsto \dfrac{x \cdot e^x}{\ln(x)}$

Answers to Practice Exercises

Answers to practice exercice 15, page 85

(1) 3

(2) ∞

(3) $1/3$

(4) $1/\sqrt{3}$

(5) ∞

(6) ∞

(7) $-\infty$

(8) ∞

(9) 0

(10) does not exist

(11) ∞

(12) does not exist

(13) does not exist

(14) ∞

(15) 0

(16) ∞

(17) 0

(18) $-3/2$

Answers to practice exercice 13, page 75

(1) 1 Use $x = 1 - u^2$.

(2) undefined – for $u > 1$ we have the square root of a negative number.

(3) $\frac{8(\sqrt{2}+1)}{15}$ Use $u = 1 + \sqrt{x}$

(4) $\frac{50}{309}$ Use $u = t^2 + 3$

(5) $\frac{195}{8}$ Use $u = x^2 + 2$

(6) $\frac{2}{45}$ Use $u = 4 - x^3$

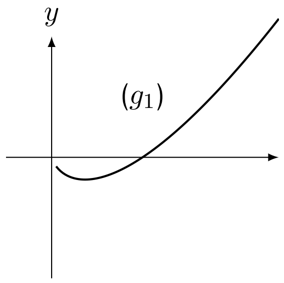
(7) $-\sqrt{6} + 2\sqrt{2}$ Use $u = 1 + \frac{1}{t}$

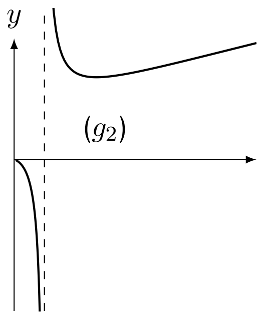
Answers to practice exercice 14, page 78
(Integration constant to be added)

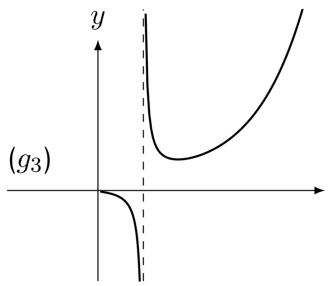
- $F_a : x \mapsto x^5 - x^2 + 4x$
- $F_b : x \mapsto \frac{1}{4}x^4 - \frac{5}{3}x^3 + \frac{3}{2}x^2 - 2x$
- $F_c : x \mapsto x^2 - x$
- $F_d : x \mapsto \frac{1}{4}x^5 - \frac{1}{4}x^3 + \frac{5}{4}x^2 + \frac{3}{2}x$
- $F_e : x \mapsto x^2 + x + \frac{1}{x}$
- $F_f : x \mapsto 3x - \frac{2}{x} + \frac{5}{2x^2}$
- $F_g : x \mapsto \frac{x^4}{4} - \frac{1}{x}$
- $F_h : x \mapsto \frac{3}{4}\sqrt[3]{x^4} + \frac{3}{2}\sqrt[3]{x^2}$
- $F_i : x \mapsto 2\sqrt{x} + \frac{2}{3}\sqrt{x^3}$
- $F_j : x \mapsto \frac{1}{3}(x+1)^3$
- $F_k : x \mapsto (3x-2)^5$
- $F_l : x \mapsto \frac{1}{8}(2x+1)^4$
- $F_m : x \mapsto -\frac{1}{12}(3-x)^{12}$
- $F_n : x \mapsto -\frac{1}{20}(3-4x)^5$
- $F_o : x \mapsto \frac{2}{9}\sqrt{(3x-2)^3}$
- $F_p : x \mapsto 2\sqrt{x-1}$
- $F_q : x \mapsto -\frac{1}{3}(3-x^2)^6$
- $F_r : x \mapsto \frac{1}{5}(x^2-3x+1)^5$

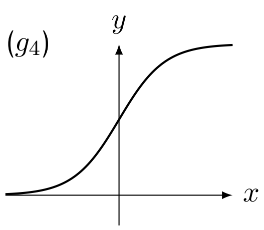
- $F_s : x \mapsto \dfrac{1}{3}(x^3 - 2x^2 + x - 3)^3$
- $F_t : x \mapsto \dfrac{2}{3}(4x^2 - 5x)^3$
- $F_u : x \mapsto \dfrac{1}{8}(3x^2 - 2x + 5)^4$
- $F_v : x \mapsto -\dfrac{1}{x^2 + 1}$
- $F_w : x \mapsto -\dfrac{1}{x^2 + x + 3}$
- $F_x : x \mapsto \dfrac{1}{3}\sqrt{(x^2+1)^3}$
- $F_y : x \mapsto 2\sqrt{9 + x^3}$
- $F_z : x \mapsto \dfrac{2}{3}(x^3 + x + 2)\sqrt{x^3 + x + 2}$
- $F_A : x \mapsto \dfrac{e^{2x}}{2}$
- $F_B : x \mapsto -\dfrac{1}{3e^{3x}}$
- $F_C : x \mapsto -\dfrac{e^{-x^2}}{2}$
- $F_D : x \mapsto -\dfrac{1}{\ln(2)}2^{-x}$
- $F_E : x \mapsto \dfrac{1}{3}(e^{2x} + 1)^{\frac{3}{2}}$
- $F_F : x \mapsto e^x(x^2 - 2x + 2)$
- $F_I : x \mapsto \dfrac{\ln(x + \frac{3}{2})}{2}$
- $F_J : x \mapsto 2x + 2\ln(x - 1)$
- $F_K : x \mapsto x - 2\ln(x + 1)$
- $F_L : x \mapsto 2x\left(\dfrac{\ln(x)^2}{2} - \ln(x) + 1\right)$
- $F_N : x \mapsto x\ln(x) - x$
- $F_O : x \mapsto x - \ln(x + 1)$
- $F_P : x \mapsto \ln(\ln(x))$

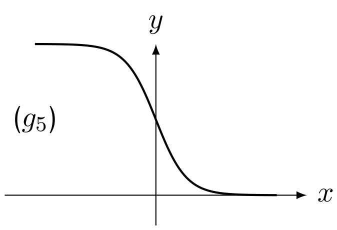
Answers to practice exercice 16, page 87

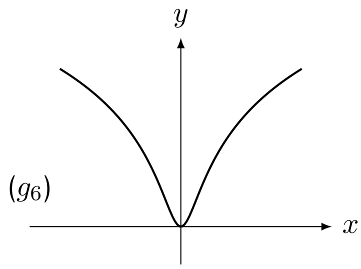


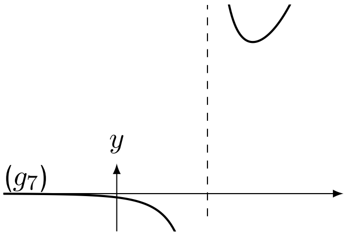








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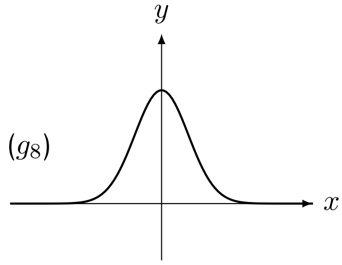
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