Information for examiners

The goal of this part is to help instructors or examiners understand a proof which uses ultrasmall numbers. This can happen when, in our school, final exams have an external juror in addition to the teacher. It can also be used to see that for the trained mathematician, the translation back to "classical" methods is quite straightforward.
This part does not explain how to write the proofs, only how to read them.

As in any book about analysis, we do not give the axioms of set theory, but state instead, properties of real numbers.

Observability

Extra Axioms – called also Principles – are used which allow to make an extra distinction within the real numbers: observability.

The intuition is that "ordinary numbers" are observable but that there are extremely small numbers (ultrasmall) which are so tiny that they are not observable. But if one zooms in to observe these tiny numbers, one can still see zero. And if such a tiny number h is aded to 2, then $2 + h$ is less observable than 2, which remains observable when 2 is observable.

Given x and y, then x is as observable as y, or y is as observable as x. They may have the same observability. Observability is transitive.

One can consider the metaphor of scales of observation.

Numbers defined without the concept of observability are observable relative to any real number: they are always observable – or standard.

Ultrasmall

Relative to any real number, there exist ultrasmall numbers, numbers which are less, in absolute value, than any observable number, yet not zero.

Relative to a, if h is ultrasmall, then $\frac{1}{h}$ is ultralarge. But then we also have that a and $a + h$ are extremely close, written $a \simeq a + h$, where the only new symbol is "$\simeq$" which reads "ultraclose": a difference which is ultrasmall or zero. Note that $a + h$ is not as observable as a.

An ultrasmall number has the "flavour" of an infinitesimal.

Context

Given a formula which has some parameters (the context), the concept of ultrasmall, ultralarge or ultraclose always refer to the whole list of parameters. If h is ultrasmall, it must be ultrasmall relative to each parameter. Thus "$\simeq$" automatically refers to all of the parameters.

Closure

Non observable numbers do not show up as results of operations if they are not introduced explicitly. Given f and a, then $f(a)$ is observable (the context being a and the parameters of f).

Observable neighbour

Any number x which is not ultralarge can be written in the form $x = a+h$ where a is observable and h is ultraclose to zero. Then a is the observable neighbour of x.

The existence of the observable neighbour is equivalent to the completeness of $\mathbb{R}$.

Limit

$\lim_{x \to a} f(x) = L$ is, here, defined by, L is observable and

$$x \simeq a \Rightarrow f(x) \simeq L.$$

Since the sum of two ultrasmall numbers is ultrasmall or zero and similar algebraic properties, it is easy to prove that the sum, resp. product, quotient of limits is the limit of the sum, resp. product, quotient.

Most students prove theorems of analysis by considering $x \simeq a$ often written as $x = a + dx$, where dx is ultrasmall (therefore not zero), positive or negative. They will perform some algebraic steps and then consider the observable part (or neighbour) of the result and check that it does not depend on dx, in particular, whether dx is positive or negative.

Translation

When reading dx or $h \simeq 0$, consider that h is a real number with the idea of it being arbitrarily small in absolute value. When we keep only the observable part of the result (if it exists), this translates to "limit when h tends to 0".

Examples

Continuity

f is continuous at a if $f(a+dx) \simeq f(a)$ whenever $dx \simeq 0$. This translates to $\lim_{h \to 0} f(a+h) = f(a)$ as usual. It is also written in a simple form: $x \simeq a \Rightarrow f(x) \simeq f(a)$

Derivative of f at a

If there is an observable value D such that for any dx we have $\frac{f(a+dx) - f(a)}{dx} \simeq D$ then $f'(a) = D$. This translates to $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = D$.

Example proof: Continuity of $f \circ g$

Assume that g is continuous at a and f is continuous at $g(a)$. Then if $x \simeq a$ we have $g(x) \simeq g(a)$ by continuity of g and $f(g(x)) \simeq f(g(a))$ by continuity of f.