ANALYSIS
using
ultrasmall numbers

Teacher's Manual

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Part I
Basic Principles

The following principles are consequences of axioms added to the classical axioms of set theory. When teaching analysis one usually does not study axioms but their consequences on real numbers. Theses will thus be considered axiomatically together with other properties of real numbers..

Definition 1
The **context** of a property, function or set is the list of parameters used in its definition. The context can be a single number.

Formally: a parameter is any number or letter which appears in a formula not linked by a quantifier ($\forall$ or $\exists$). Quantifiers are probably best not used at introductory level. A definition which has its pedagogical advantages is" parameters are what the statement is about". Thus for the continuity of f at a, the statement is about f (its parameters) and a, not about any auxiliary or dummy variables – whether ε and δ in the classical approach or the ultrasmall increments used here.

Observability Principle

- A number is observable relative to a context if it is observable relative to at least one parameter of the context.
- Every number is observable relative to some context.
- Two numbers a and b will always have a common context. If a is not observable relative to b, then b will be observable relative to a.

The word "observable", by convention, refers to a context. Informally: the context is the parameters, sets and functions the statement is about. Therefore to determine the context of a statement, one must be able to understand it and describe what it says and about what it says something.

If a number is observable whenever any other number is observable, we say that it is *always* observable. We may also say that it is *standard*.

Closure Principle
Numbers defined without reference to observability are always observable.
If a number satisfies a given property, then there is an observable number satisfying that property

(Observability here being given by the property).

The closure principle tells us that all "familiar" numbers such as $1;\ 3;\ 10^{10};\ \sqrt{2}$ or π are always observable

It also yields that if a number is calculated using some parameters, the resulting number will be observable. Non observable results do not show up unless explicitly summoned.

The problem of induction

⚠ Note for teachers: it is not possible to proceed by induction to show that all numbers are observable.

The reader familiar with induction may conclude that since 1 is always observable then $2 = 1 + 1$ is observable – which is true by closure, and that if n is always observable, then $n + 1$ is also always observable – which is also true also by closure. But it would be false to conclude that this proves that all natural numbers are always observable. They are not: there *are* ultralarge integers. Induction is a property which is valid for classical statements which do not use the concept of observability. We will see that we can extend induction to "contextual" statements. The statement "is always observable" is clearly not a classical statement so it may not be used in induction nor to define a set: there is no set containing all and only numbers which are always observable. See page 5 for an explanation of which statements can be used to define sets and induction.

This aspect of induction is probably the most troubling to mathematician newcomers, but for students with no prior knowledge, we have observed that it is not a problem.

Example

Let $f : x \mapsto x^2 + 3$, The parameters of f are 2 and 3 which are always observable. The number $f(4)$ is thus also always observable. In general $f(x)$ is as observable as x.

Definition 2

A real number is **ultrasmall** if it is non zero and smaller in absolute value than any strictly positive observable number

This definition makes an implicit reference to a context which therefore must be determined before an ultrasmall number is referred to.

Principle of ultrasmallness
Whatever the number x, there exist ultrasmall real numbers relative to x.

Note in particular that if ε is ultrasmall relative to some context containing a then ε is not observable neither is $a + \varepsilon$.

Definition 3

A real number is **ultralarge** if it is larger in absolute value than any strictly positive observable number

⚠ Note that 0 is not ultrasmall. This can be justified by observing that the reciprocal of an ultrasmall is ultralarge and 0 has no reciprocal.

⚠ Note the asymmetry: if h is ultrasmall relative to x, then it is not observable. But then x is observable relative to h, hence x **is not** ultralarge relative to h.

Definition 4

Let a, b be real numbers. We say that a is **ultraclose** to b (relative to some context), written

$$a \simeq b,$$

if $b - a$ is ultrasmall or if $a = b$.

In particular, $x \simeq 0$ if x is ultrasmall or zero.

Contextual Notation
The only acceptable properties are those that do not refer to observability ("classical" definitions) or those that use the symbol "$\simeq$".

As one would expect (see below for a proof) the reciprocal of an ultralarge is an ultrasmall, thus n is ultralarge is characterised by stating that $1/n \simeq 0$.

In class, it is possible to replace the second part of the closure principle by one of its consequences, using contextual notation:

$$\boxed{f(a) \quad \text{is observable}}$$

This refers to the context, by the word "observable". The only parameters of this property are f and a.

A context is *extended* if parameters are added to the list.

Stability Principle
A property is true if and only if the context is replaced by an extended context.

This principle ensure, for example, that if f and g are functions, then the derivatives $f'(a)$ and $g'(a)$ remain the same even if $f'(a)$ is calculated using the additional parameters of g. For all functions given by explicit rules it is seen by inspection. Formally, it is a consequence of stability.

"$\simeq$" is the only new symbol introduced.

Why ultralarge rather than infinity?

The classical definition of integers being finite cardinalities remains true hence ultralarge integers cannot be infinite. They are huge, very huge, but not infinite.
Consequently, their reciprocals will be ultrasmall. These numbers are real numbers.
This approach contradicts no classical mathematical statement but by using an extra distinction, it can make statements that cannot be expressed without the concept of observability.

What happens if contextual notation is not used?

The "$\simeq$" symbol is defined only relative to some context. Referring in a non contextual way would require to invent a new notation. The reader may find it amusing to try and invent such a notation of their own and see "pathological" objects. For instance, something that looks like a function, everywhere horizontal, that looks continuous and yet is globally increasing! These are said to be "external" objects and they do not necessarily satisfy classical theorems. We will not address this issue further except to say that contextual notation guarantees that objects they define are really sets, functions or properties in the usual sense and can be used in induction.

Relative to some context: If $a \simeq b$ then a and b are said to be neighbours. If a is a neighbour of b and is observable then a is the observable neighbour of b.

Principle of the observable neighbour
Relative to a context, any real number x which is not ultralarge can be written in the form $a + h$ – where a is observable and $h \simeq 0$.

(We also say "observable part".)
a is the **observable part** of x.

This principle is very close to the completeness of the reals.

Exercise 1 (answer page 20)
Using the principles and definitions, show that (relative to a given context)

(1) If ε is ultrasmall, then $\frac{1}{\varepsilon}$ is ultralarge.

(2) If M is ultralarge then $\frac{1}{M}$ is ultrasmall.

Rule 1
Given a context. Let a be observable and non zero and h ultrasmall and $\varepsilon \simeq 0$. Then

(1) $a \cdot h \simeq 0$

(2) $\dfrac{a}{h}$ is ultralarge.

(3) $\varepsilon \cdot h \simeq 0$

(4) $\varepsilon + h \simeq 0$

Exercise 2 (answer page 20) Prove rule 1.

Rule 2
Given a context. Let a and b be observable and x and y be such that $a \simeq x$ and $b \simeq y$. Then

(1) $a \pm b \simeq x \pm y$.

(2) $a \cdot b \simeq x \cdot y$.

(3) If $b \neq 0$ then $\dfrac{a}{b} \simeq \dfrac{x}{y}$.

(4) $a \simeq b \Rightarrow a = b$.

Exercise 3 (answer page 20) Prove rule 2.

We refer to these rules as "ultracalculus". The proofs are algebraic and a good training for students to learn how to work with definitions.

In class this requires some attention, but then, this being done, the rules equivalent to adding, multiplying and dividing limits are immediate.

A consequence of (4) is that the observable part is unique. If $a \simeq x \simeq b$ with a and b observable, then $a = b$. This is equivalent to the uniqueness of the limit!

The existence of the observable neighbour is not guaranteed in $\mathbb{Q}$. Let x be a rational ultraclose to $\sqrt{2}$ (for ultralarge whole number N, take the first N digits of $\sqrt{2}$). $\sqrt{2}$ is standard, by closure: it is the unique positive solution of $x^2 = 2$ whose only parameter is 2 which is standard. Since $\sqrt{2}$ is standard and not rational, the observable neighbour of x is not in $\mathbb{Q}$.

Note on the presentation to students:

It is not necessary to prove all results given here before studying the derivative. With explicit rules for functions, it is possible to prove them when they are needed.

Part II

The derivative

We will not show the proofs of all theorems about derivatives but we hope to show enough that the reader can perform the remaining proofs as exercises.

The definition of the derivative at a point requires that the funciton be defined at least on an open interval $]b, c[$ containing a. Since the domain is determined by f (closure), it is observable and we can always suppose that b, c are observable.

Since x is the independant variable, its increment can always be chosen to be non zero. It can be positive or negative. We write dx for this ultrasmall (non zero) increment.

Definition 5
Let f be a real function defined on an open interval containing a.
We say that f is differentiable at a if there is an observable number D such that for any ultrasmall dx we have

$$\frac{f(a+dx)-f(a)}{dx} \simeq D$$

We write $D = f'(a)$, the derivative of f at a.

(The context is given by f and a)

- The result must not depend on dx
- When it exists, the derivative is the observable neighbour of $(f(a+dx)-f(a))/dx$.

Example
Let

$$f : x \mapsto x^2 + 3x$$

For the derivative at $x = 5$. The parameters are 2,3 and 5 (the context). Let dx be ultrasmall. Then

$$\frac{f(5+dx)-f(5)}{dx} = \frac{((5+dx)^2+3(5+dx))-(25+15)}{dx} = \frac{10dx+3dx+dx^2}{dx} = 10+3+dx.$$

Then $13 + dx \simeq 13$ which is observable and does not depend on dx, hence it is the derivative.

⚠ The same proof could be done with x in general. Then x is part of the context. For those familiar with other forms of nonstandard calculus: note that here we could have directly shown that $f'(x) = 2x + 3$ for all x whether always observable or less observable. This is one of the main features of this approach.

⚠ Note that it is possible to be reasonably "careless" about the context. When the formula is expanded, dx must be ultrasmall relative all other terms in the expansion.

Example
Let

$$g : x \mapsto |x|$$

at 0. *Let* dx *be ultrasmall. If* $dx > 0$, *then*

$$\frac{g(0+dx)-g(0)}{dx} = \frac{g(dx)-g(0)}{dx} = \frac{dx-0}{dx} = \frac{dx}{dx} = 1.$$

But if $dx < 0$, *then*

$$\frac{g(0+dx)-g(0)}{dx} = \frac{g(dx)-g(0)}{dx} = \frac{-dx-0}{dx} = \frac{-dx}{dx} = -1.$$

There is thus no unique real number satisfying the condition independently of dx. *The conclusion is that the derivative of* g *does not exist for* $x = 0$.

The modulus function is defined with no reference to observability, it is thus a function which is always observable. But in fact, one can ignore this and simply use that dx is ultrasmall relative to the function and 0 without further specification.

The line tangent to a curve can now be defined using the derivative (a straight line passing through $\langle x_0, f(x_0)\rangle$ and having slope $f'(x_0)$). This is in fact simpler than defining the derivative as the slope of the tangent with the difficulty of defining the tangent before the derivative.

Differentiation rules

Let dx be ultrasmall relative to a and f. We write

$$\Delta f(a) = f(a+dx) - f(a).$$

Then

$$\frac{\Delta f(x)}{dx} \simeq f'(x).$$

And also $f(x+dx) = f(x) + \Delta f(x)$.

We now show without further comment the proofs of some the usual rules of differentiation and leave the others as exercises.

We do not specify the context explicitly since it should by clear now that it is the list of parameters. (dx is not a parameter but a dummy variable, as are the ε and δ of classical proofs: the context is "what we are talking about".) dx is always chosen ultrasmall, hence we do not specify it every time.

Theorem 1
If $f'(a)$ exists, then $\Delta f(a) \simeq 0$.

Proof: $\Delta f(a) = \frac{\Delta f(a)}{dx} \cdot dx \simeq f'(a) \cdot dx \simeq 0$

□

Alternative proof: $\Delta f(a) = f(a+dx) - f(a) = f'(a) \cdot dx + \varepsilon \cdot dx$ for $\varepsilon \simeq 0$. It is then immediate that $\Delta f(a) \simeq 0$.

This shows that differentiability implies continuity, but since here, we choose to study derivatives before continuity, the concept is not mentioned.

Theorem 2
Let f and g be functions differentiable at a. Then the function $f \cdot g$ is differentiable at a and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a).$$

Proof:

$$\begin{aligned}
\frac{\Delta(f \cdot g)(a)}{dx} &= \frac{f(a+dx) \cdot g(a+dx) - f(a) \cdot g(a)}{dx} \\
&= \frac{\Big(f(a) + \Delta f(a)\Big) \cdot \Big(g(a) + \Delta g(a)\Big) - f(a) \cdot g(a)}{dx} \\
&= \frac{f(a) \cdot \Delta g(a) + \Delta f(a) \cdot g(a) + \Delta f(a) \cdot \Delta g(a)}{dx} \\
&= f(a) \cdot \frac{\Delta g(a)}{dx} + \frac{\Delta f(a)}{dx} \cdot g(a) + \frac{\Delta f(a) \cdot \Delta g(a)}{dx} \\
&\simeq f'(a) \cdot g(a) + f(a) \cdot g'(a),
\end{aligned}$$

since, in particular, $\dfrac{\Delta f(a) \cdot \Delta g(a)}{dx} \simeq f'(a) \cdot \Delta g(a) \simeq 0$ ($\Delta g(a) \simeq 0$ by theorem 1 and $f'(a)$ is observable by its definition, and observable$\times$ ultrasmall is ultrasmall, by rule 1).

Since $f'(a) \cdot g(a) + f(a) \cdot g'(a)$ is observable, it is the derivative.

$\square$

Stability is in fact hidden in the proof. If $g(a)$ is not as observable as $f(a)$ then the context for f' is extended to contain also $g(a)$ and this becomes the general context. In class, almost all functions are always observable so this subtlety is not really an issue and in the case of explicitly given functions it is true by inspection.

These proofs are not technically very different from the classical proofs. Cognitively they are very different. There is no "taking the limit". There is no need to justify dividing by h not zero then letting it vanish. There is no need to explain that the limit is the number that we approach without necessarily reaching it. We simple compute and take the part which is observable.

Theorem 3 (Chain Rule)
Let g be a function differentiable at a and f a function differentiable at $g(a)$. Then the function $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Proof: We consider two cases (1) $\Delta g(a) = 0$ and (2) $\Delta g(a) \neq 0$.

(1) If $\Delta g(a) \neq 0$, then

$$\begin{aligned}
\frac{f(g(a+dx)) - f(g(a))}{dx} &= \frac{f(g(a) + \Delta g(a)) - f(g(a))}{dx} \\
&= \frac{f(g(a) + \Delta g(a)) - f(g(a))}{\Delta g(a)} \cdot \frac{\Delta g(a)}{dx} \\
&\simeq f'(g(a)) \cdot g'(a)
\end{aligned}$$

since f is differentiable at $g(a)$ and since g is differentiable a we have $\Delta g(a) \simeq 0$.

The proof may be easier to read if we use Leibniz' notation, replacing $g(a)$ by y and $\Delta g(a)$ by Δy, then

$$\frac{\Delta f(y)}{dx} = \frac{\Delta f(y)}{\Delta y} \cdot \frac{\Delta y}{dx} \simeq f'(y) \cdot y'$$

(2) If $\Delta g(a) = 0$ then $g(a+dx) = g(a)$ and $g'(a) = 0$, therefore $\dfrac{f(g(a+dx)) - f(g(a))}{dx} = 0$. And $f(g(a))' = f'(g(a)) \cdot g'(a)$.

□

Definition 6
If f is differentiable at a, the quantity $f'(a) \cdot dx$ is noted $df(a)$: it is the **differential** of f at a.

Then we have

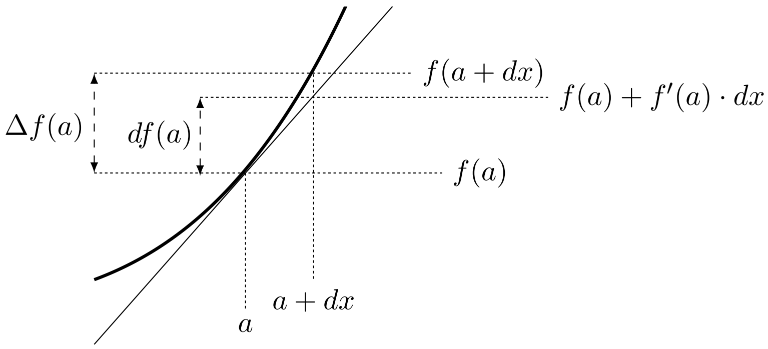
$$\frac{df(a)}{dx} = f'(a)$$

which is an equality. The expression $df(a)/dx$ here, really is a quotient.

⚠ Note the difference between Δy and dy i.e., between the variation and the differential. We have $\dfrac{\Delta y}{dx} \simeq y'$ hence $\dfrac{\Delta y}{dx} = y' + \varepsilon$ (for $\varepsilon \simeq 0$). Thus

$$\begin{aligned} \Delta y &= y' \cdot dx + \varepsilon \cdot dx \\ &= dy + \varepsilon \cdot dx \end{aligned}$$

The differential is the variation along the tangent line.



Theorem 4 (de l'Hospital's Rule for $0/0$ – simple form)
Let f and g be functions differentiable at a. Suppose that $f(a) = g(a) = 0$ but that $g'(a) \neq 0$. Then

$$\frac{f(a+dx)}{g(a+dx)} \simeq \frac{f'(a)}{g'(a)}$$

Exercise 4 (answer page VI) Prove theorem 4.

Note that this is a simple form of the rule. It also holds for $\frac{\simeq 0}{\simeq 0}$ or $\frac{\infty}{\infty}$ situations which are harder to prove. The extension to these cases is admitted without proof in class[1])

Part III
Continuity

Definition 7
Let f be a function defined on an open interval containing a. We say that f is **continuous at** a if

$$f(x) \simeq f(a) \quad \text{for every } x \simeq a.$$

This is a property of f and a hence this is the context.
Alternatively: f is continuous at a if

$$f(a + dx) \simeq f(a)$$

or still

$$\Delta f(a) \simeq 0$$

If f is differentiable at a then f is continuous at a. This is a restatement of theorem 1.

Theorem 5
Let f and g be two functions continuous at a. Then

(1) $f \pm g$ is continuous at a.

(2) $f \cdot g$ is continuous at a.

(3) $\dfrac{f}{g}$ is continuous at a if $g(a) \neq 0$.

Proof: This theorem is a direct application of the rules that $x \simeq a$ and $y \simeq b$ imply $x \cdot y \simeq a \cdot b$ et $x + y \simeq a + b$ and $x/y \simeq a/b$ (rule 2).

□

⚠ In the classroom, it may be better to systematically use the definitions rather than properties previously shown. For the product this gives:

Proof: [of (2)]

$$\begin{aligned} f(a+dx)\cdot g(a+dx) &= (f(a)+\Delta f(a))(g(a)+\Delta g(a)) \\ &= f(a)\cdot g(a) + f(a)\cdot \Delta g(a) + \Delta f(a)\cdot g(a) + \Delta f(a)\cdot \Delta g(a) \\ &\simeq f(a)\cdot g(a) \end{aligned}$$

since f and g are continuous and hence $\Delta f(a) \simeq 0$ and $\Delta g(a) \simeq 0$.

□

Theorem 6
Suppose that g is continuous at a and that f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.

[1] We use here, for teachers, the ∞ symbol as shorthand for "ultralarge". See below for the use in class.

Proof: Let $x \simeq a$. Then $g(x) \simeq g(a)$ by continuity of g at a so $f(g(x)) \simeq f(g(a))$ by continuity of f at $g(a)$.

□

Part IV
Limit

The concept of limit is not necessary to study calculus using ultrasmall numbers. Nonetheless there are good reasons to introduce it. One straightforward reason is that studying the limit is part of the syllabus. Furthermore, students will see limits used in publications and must be able to transpose the definition. Last: the writing of the limit is a convenient shorthand.

Continuity being defined without reference to the limit, we can define the limit in terms of continuity without circularity.

The limit of f at a is the value that f *should* take to be continuous at a.

Exercise 5 (Answer page 21.)
Write the definition of the limit, using the definition given above and the definition of continuity (definition 7).

Example
Consider the function

$$f : x \mapsto \frac{2x^2 - 7x + 3}{x - 3}, \qquad \textit{with } Dom(f) = \mathbb{R} \setminus \{3\},$$

and its limit at $a = 3$. The function f is defined around 3. Let $x \simeq 3$ ($x \neq 3$) and write $x = 3 + dx$. But then

$$f(x) = \frac{2(3 + dx)^2 - 7(3 + dx) + 3}{(3 + dx) - 3} = \frac{5dx + dx^2}{dx} = 5 + dx \simeq 5.$$

Since 5 does not depend on dx and is ultraclose to $f(x)$, it is the limit.

The classical property that f is not defined at $x = 3$ remains true.

The limit is unique. Suppose we choose x even closer to 3, say $x = 3 + \varepsilon$, where ε is ultrasmall relative to the dx. Then similarly

$$f(x) = \frac{2(3 + \varepsilon)^2 - 7(3 + \varepsilon) + 3}{(3 + \varepsilon) - 3} = 5 + \varepsilon,$$

where the difference between $f(x)$ with 5 is ε, ultrasmall relative to dx! This is in fact a very general consequence of stability

$$x \simeq a \ (x \neq a) \quad \text{implies} \quad f(x) \simeq L$$

is true if and only if it is true relative to an extended context.

The context is not important provided all parameters are observable. It may contain extra parameters.

Theorem 7
If the limit of f at a exists then it is unique.

Proof: Suppose that L_1 and L_2 are observable and such that

$$f(x) \simeq L_1 \quad \text{and} \quad f(x) \simeq L_2$$

whenever $x \simeq a$ $(x \neq a)$.
But then

$$L_1 \simeq L_2 \quad \text{are both observable.}$$

therefore $L_1 = L_2$ (by rule 2.4)

□

We write

$$\lim_{x \to a} f(x) = L$$

if the limit of f at a is L.

⚠ In class the introduction of the limit almost always causes some instability. It seems that the symbol"=" where they expect "$\simeq$" poses a problem. The usual difficulties of the concept reappear. Nonetheless, with the formalism of ultrasmall numbers, it is possible to insist that it is only a shortcut and that every time a limit is encountered, it may be replaced by its definition, thus reintroducing the "$\simeq$" symbol.

We introduce the symbol ∞:

Definition 8
Let a be a real number and f a function defined around a. We write

$$\lim_{x \to a} f(x) = +\infty$$

if $f(x)$ is ultralarge positive whenever $x \simeq a$ $(x \neq a)$
Similarly for $-\infty$.

⚠ Even in this approach, the symbol ∞ does **not** denote a number.

We sometimes use $x \simeq \infty$ to indicate that x is ultralarge, insisting that since ∞ is not a number, x is not really ultraclose to it. It is useful to describe, for example, a vertical asymptote at $x = a$ by writing $x \simeq a \Rightarrow f(x) \simeq \infty$ and a horizontal asymptote is defined by switching terms: $x \simeq \infty \Rightarrow f(x) \simeq a$.

Part V
Intermediate Value Theorem

This theorem – in high school – is often given without proof. We give it here so that the instructor may have a global view.

For local properties (derivatives, continuity) the method is to look at ultraclose neighbours to determine an approximation of the required value. The observable part being the exact value. For global properties (Intermediate Value, Extreme value, Integral) the method is to divide the interval into an ultralarge numbers of pieces, find the best approximation and use its observable part.

For $f(a) < 0 < f(b)$ we search for a c in $[a;b]$ such that $f(c) \simeq 0$. The context is a, b and f. We first choose an ultralarge whole number N. Then $dx = (b-a)/N$ is ultrasmall. We consider the $N+1$ points $x_i = a + i \cdot dx$, for $i = 0, \ldots, N$. Then we find an "ultragood" approximation to the required number. The observable neighbour of this turns out to be the number having the property.

Theorem 8 (Intermediate Value)
Let f be a function continuous on $[a;b]$ such that $f(a) < 0 < f(b)$. Then there is a $c \in [a;b]$ such that $f(c) = 0$.

Proof: (The context is f, a, b and d.)

Let N be a positive ultralarge integer and let $dx = (b-a)/N$. Then $dx \simeq 0$. Consider $x_i = a + i \cdot dx$, for $i = 0, \ldots, N$ (hence $x_0 = a$ and $x_N = b$). Since it is a finite collection, there is a first index j such that $f(x_{j+1}) \geq 0$. Then we have

$$f(x_j) \leq 0 \leq f(x_{j+1}).$$

Let c be the observable neighbour of x_j (it exists since x_j is bounded by observable a and b). Then $x_j \simeq c$. Furthermore, $c \in [a;b]$ and $c \simeq x_{j+1}$ because $x_j \simeq x_{j+1}$. By continuity of f at c we have

$$f(c) \simeq f(x_j) < 0 \quad \text{and} \quad f(c) \simeq f(x_{j+1}) \geq 0.$$

We deduce that

$$f(c) \simeq 0.$$

But $f(c)$ is observable by closure and also 0, so

$$f(c) = 0.$$

□

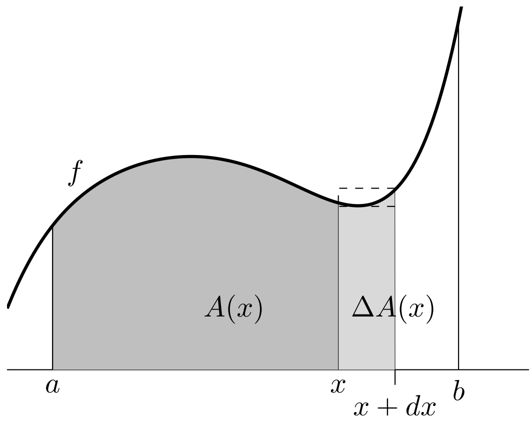
Part VI
The Integral

We consider two approaches to the integral. First assuming that an area function exists[2], then by summing an ultralarge number of ultrasmall values.

Consider a non negative function f continuous on $[a;b]$. Let $A(x)$ be the area between the function and the x-axis between a and x.

The variation of the area between x and $x + dx$ is noted $\Delta A(x)$.

[2] Formally, we should first define what such an area is. With students, as usual, we may assume, without further discussion or measure theory, that the area is a well defined concept.



Theorem 9
Let f be a non negative function continuous on $[a; b]$. Then the function

$$A : x \mapsto A(x),$$

where $A(x)$ is the area under the curve between a and x satisfies the two following properties:

(1) $A'(x) = f(x)$, for every $x \in [a; b]$.

(2) $A(a) = 0$.

Proof: (2) is obvious. We show (1).

The context is a, f and x. Let dx be ultrasmall and positive. Since f is continuous on $[x; x + dx]$ it attains its maximum and minimum on $[x; x + dx]$. Note $(x_M, f(x_M))$ for the maximum and $(x_m, f(x_m))$ for the minimum. Then

$$f(x_m) \cdot dx \leq \Delta A(x) \leq f(x_M) \cdot dx.$$

Therefore

$$f(x_m) \leq \frac{\Delta A(x)}{dx} \leq f(x_M).$$

As f is continuous at x (which is part of the context hence observable) and $x \simeq x_M$ and $x \simeq x_m$ we have $f(x) \simeq f(x_M)$ and $f(x) \simeq f(x_m)$ (hence also $f(x_M) \simeq f(x_m)$), this implies that

$$\frac{\Delta A(x)}{dx} \simeq f(x).$$

The same result follows if dx is negative and

$$A'(x) = f(x),$$

because $f(x)$ is observable.

□

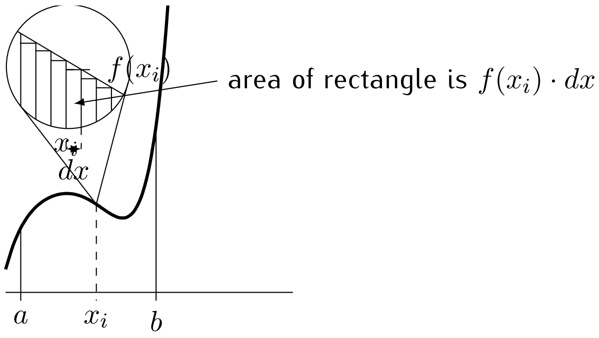
For more advanced students, we see the following.

The area can also be calculated by adding an ultralarge number of ulltrathin slices and showing the the total error (the little almost triangular pieces on top) amount to an ultrasmall error.

We assume here that f is continuous and that we know an antiderivative $F'(x) = f(x)$ for all x of $[a, b]$. Let N be ultralarge and write $dx = (b - a)/N$ and $x_i = a + i \cdot dx$, for $i = 0, \ldots, N$.

Each slice given above on interval $[x_i; x_i + dx]$ has an area approximated by a rectangle with base dx and height approximately $f(x_i)$ Then the total area is approached by

$$\sum_{i=0}^{N-1} f(x_i) \cdot dx.$$



Definition 9
Let f be a function defined on $[a; b]$. Let n be an ultralarge whole number. Let $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$ for $i = 0, \ldots, n$. We say that f **is integrable on** $[a; b]$ if there is an observable real number I such that

$$\left(\sum_{i=0}^{n-1} f(x_i) \cdot \Delta x\right) \simeq I.$$

If such a number exists, we call it the '**integral of** f **between** a **and** b written

$$\int_a^b f(x) \cdot dx.$$

By closure, the integral is observable whenever a, b and f are observable.

Theorem 10 (Fundamental Theorem of Analysis)
Let f be a function continuous on $[a; b]$. Let F be an antiderivative of f on $[a; b]$. Then

$$\int_a^b f(x) \cdot dx = F(b) - F(a).$$

Proof:
Let N be a positive ultralarge integer[3] let $dx = (b-a)/N$ and $x_i = a + i \cdot dx$ for $i = 0, \ldots, N$. We write $F(b) - F(a)$ as a telescoping sum:

$$F(b) - F(a) = \sum_{i=0}^{N-1} F(x_{i+1}) - F(x_i) = \sum_{i=0}^{N-1} F(x_i + dx) - F(x_i).$$

By the mean value theorem, there is an x in $[x_i, x_{i+1}]$ such that $F(x_{i+1}) - F(x_i) = F'(x) \cdot dx$.
Let c be the observable neighbour of x, then $F'(x) \simeq F'(c) \simeq F'(x_i)$. Hence $F'(x) \simeq F'(x_i) + \varepsilon_i$ and

$$F(x_{i+1}) - F(x_i) = F'(x_i) \cdot dx + \varepsilon_i \cdot dx = f(x_i) \cdot dx + \varepsilon_i \cdot dx.$$

[3]The context is a, b and f.

Therefore we have

$$F(b) - F(a) = \sum_{i=0}^{N-1} f(x_i) \cdot dx + \sum_{i=0}^{N-1} \varepsilon_i \cdot dx.$$

We now show that

$$\sum_{i=0}^{N-1} \varepsilon_i \cdot dx \simeq 0.$$

Let ε be $\max\{\varepsilon_i\}$ Then

$$\sum_{i=0}^{N-1} \varepsilon_i \cdot dx \leq \sum_{i=0}^{N-1} \varepsilon \cdot dx = \varepsilon \cdot \sum_{i=0}^{N-1} dx = \varepsilon \cdot (b-a) \simeq 0$$

Hence

$$F(b) - F(a) \simeq \sum_{i=0}^{N-1} f(x_i) \cdot dx,$$

but as $F(b) - F(a)$ is observable f is thus integrable and

$$F(b) - F(a) = \int_a^b f(x) \cdot dx.$$

$\square$

Integration rules

Integration rules are shown as usual by assuming the existence of antiderivatives. We do not show these here.

We do show, however integration by variable substitution by an example. It is an almost hidden use of the chain rule and is given as a method rather than a theorem.

Example
Consider

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx.$$

Let $u = 1 + \sqrt{x}$*. (Therefore* $u - 1 = \sqrt{x}$*)*

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2(u-1)}$$

hence

$$dx = 2 \cdot (u-1) \cdot du.$$

Then $\sqrt{1+\sqrt{x}} = \sqrt{u}$ *and* $\sqrt{1+\sqrt{x}} \cdot dx = \sqrt{u} \cdot 2 \cdot (u-1) \cdot du$

If $x = 0$ *then* $u = 1$ *and if* $x = 1$ *then* $u = 2$.

x	u
0	*1*
1	*2*

Replacing all terms we get

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx = 2\int_1^2 \sqrt{u} \cdot (u-1) \cdot du = 2\int_1^2 \left(u^{3/2} - u^{1/2}\right) \cdot du$$

so that

$$2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^2 = \frac{8+8\sqrt{2}}{15}.$$

Since $u = 1 + \sqrt{x}$, it is also possible to go back to the original variable x and find an antiderivative.

$$\int \sqrt{1+\sqrt{x}} \cdot dx = \frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5 - \frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3 + C.$$

Answers to exercises

Answer to exercise 1, page 6

Fix a context.

(1) By contradiction: assume ε ultrasmall and positive and that $\frac{1}{\varepsilon}$ is not ultralarge, i.e., there is an observable a such that $\frac{1}{\varepsilon} < a$. Then $\frac{1}{a} < \varepsilon$, but by closure, $1/a$ is observable (and positive) contradictibg that ε is ultrasmall.

(2) By contradiction as above, assuming there is an observable non zero b such that $b < \frac{1}{M}$ which leads to $M < \frac{1}{b}$. The last term is observable by closure, whcih contradicts that M is ultralarge.

We suggest to stop a little while on this proof. It shows the power of the closure principle. This principle and the observable neighbour principle are the fundamental tools in this approach. Stability is also fundamental but is often used without noticing it.

Answer to exercise 2, page 7

Proof: Without loss of generality we consider all terms to be positive.

(1) By contradiction. Suppose that $a \cdot h \geq b$ for some observable $b \neq 0$. Then $h \geq \frac{b}{a}$. By closure b/a is observable and this contradicts that h is ultrasmall.

(2) Suppose $\frac{a}{h} < b$ hence $a < b \cdot h$. If b were observable, $b \cdot h$ would be ultrasmall (see above)and could not be grater than some observable number. If b is not observable but smaller than some observable c, then $\frac{a}{h} < b < c \Rightarrow a < b \cdot h < c \cdot h$ and the same conclusion follows. Hence $\frac{a}{h}$ cannot be less than an observable and thus is ultralarge.

(3) For any non zero observable a we have $0 < h < a$ hence $0 < \varepsilon \cdot h < \varepsilon \cdot a \simeq 0$.

(4) $\varepsilon + h \leq 2 \cdot \underbrace{\max\{\varepsilon, h\}}_{\simeq 0} \simeq 0$ by (1).

□

Answer to exercise 3, page 7

Proof:

(1) By the observable neighbour principle we can write$x = a + \varepsilon$ and $y = b + \delta$. With $\varepsilon, h \simeq 0$.

$$x \pm y = (a + \varepsilon) \pm (b + \delta) = (a \pm b) + \underbrace{(\varepsilon \pm \delta)}_{\simeq 0},$$

therefore $a \pm b \simeq x \pm y$.

(2) Similarly for the product:

$$x \cdot y = (a + \varepsilon) \cdot (b + \delta) = a \cdot b + \underbrace{a \cdot \delta}_{\simeq 0} + \underbrace{b \cdot \varepsilon}_{\simeq 0} + \underbrace{\varepsilon \cdot \delta}_{\simeq 0}$$

by rule 1 hence $a \cdot b \simeq x \cdot y$.

(3) We first show that $\frac{1}{b} \simeq \frac{1}{y}$.

Let $k = \dfrac{1}{b} - \dfrac{1}{y} = \dfrac{y-b}{b \cdot y}$ from which $y - b = k \cdot b \cdot y$. But $y - b \simeq 0$ hence $k \cdot b \cdot y \simeq 0$. Since $b \cdot y \simeq b^2 \not\simeq 0$ we necessarily have $k \simeq 0$.

The general rule is obtained by combining previous results.

$$\frac{a}{b} = a \cdot \frac{1}{b} \simeq x \cdot \frac{1}{y}.$$

(4) If $a \simeq b$ then $a - b \simeq 0$. Since a and b are observable, their difference is observable. Therefore it cannot be ultrasmall, hence it is zero.

$\square$

Answer to exercise 4, page 11

Proof: The context is a, f and g. Let $x \simeq a$ with $x \neq a$. We write $x = a + dx$). Then

$$\frac{f(a+dx)}{g(a+dx)} = \frac{f(a) + \Delta f(a)}{g(a) + \Delta g(a)} = \frac{\Delta f(a)}{\Delta g(a)} = \frac{\dfrac{\Delta f(a)}{dx}}{\dfrac{\Delta g(a)}{dx}} \simeq \frac{f'(a)}{g'(a)}.$$

$\square$

It is also possible to use $f(a+dx) = f(a) + f'(a) \cdot dx + \varepsilon \cdot dx$ and $g(a+dx) = g(a) + g'(a) \cdot dx + \delta \cdot dx$, hence

$$\frac{f(a+dx)}{g(a+dx)} = \frac{f'(a) \cdot dx + \varepsilon \cdot dx}{g'(a) \cdot dx + \delta \cdot dx} = \frac{f'(a) + \varepsilon}{g'(a) + \delta} \simeq \frac{f'(a)}{g'(a)}$$

Answer to exercise 5, page 13

Definition 10
Let f be function defined around a. We say that the **limit of f at a exists** if there is an observable real number L such that

$$f(x) \simeq L \quad \text{whenever } x \simeq a \ (x \neq a).$$