RBST is conservative over ZFC.

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Introduction

The goal of these notes is to give a self-contained proof of the fact that **RBST**, as formulated in [6], is a conservative extension of **ZFC**. Some familiarity with ultrafilters and ultraproducts would be very helpful, even though we give all the definitions and prove the needed results here. The one exception is the proof of the existence of good ultrafilters, for which we refer to [2].

Bounded Set Theory

The nonstandard set theory **BST** (*Bounded Set Theory*) is formulated in a language with a binary predicate symbol $\in$ and a unary predicate symbol $\mathbf{S}$.

Let $\mathcal{P}$ be any $\in$-statement. Then $\mathcal{P}^{\mathbf{S}}$ denotes the *relativization of* $\mathcal{P}$ *to* $\mathbf{S}$, i.e., the statement obtained from $\mathcal{P}$ by restricting all quantifiers to $\mathbf{S}$. In more detail, this means replacing each occurrence of the existential quantifier $\exists$ in $\mathcal{P}$ by $\exists^{\mathbf{S}}$, where $(\exists^{\mathbf{S}})\ldots$ is shorthand for $(\exists x)(\mathbf{S}(x) \wedge \ldots)$, and replacing each occurrence of the universal quantifier $\forall$ by $\forall^{\mathbf{S}}$, where $(\forall^{\mathbf{S}})\ldots$ is shorthand for $(\forall x)(\mathbf{S}(x) \rightarrow \ldots)$.

The notation $\bar{x}$ is used as shorthand for a list of variables $x_1, \ldots, x_k$.

The axioms of **BST** are:

- **ZFC in S:**
$$\mathcal{P}^{\mathbf{S}}, \text{ where } \mathcal{P} \text{ is any axiom of } \mathbf{ZFC}.$$

- **Boundedness:**
$$(\forall x)(\exists^{\mathbf{S}} y)(x \in y).$$

- **Transfer:**
$$(\forall^{\mathbf{S}} x_1)\ldots(\forall^{\mathbf{S}} x_k)\ (\mathcal{P}^{\mathbf{S}}(x_1,\ldots,x_k) \Leftrightarrow \mathcal{P}(x_1,\ldots,x_k))$$
where $\mathcal{P}(x_1,\ldots,x_k)$ is any statement in the $\in$-language.

- **Standardization:**
$$(\forall \bar{x})(\forall x)(\exists^{\mathbf{S}} y)(\forall^{\mathbf{S}} z)(z \in y \Leftrightarrow z \in x\ \wedge\ \mathcal{P}(z,x,\bar{x}))$$
where $\mathcal{P}(z,x,\bar{x})$ is any statement in the $\in$-$\mathbf{S}$-language.

- **Bounded Idealization:**

$$(\forall \bar{x})(\forall^{\mathbf{S}} A)[(\forall^{\mathbf{S}} a \in \mathcal{P}^{\mathbf{fin}}(A))(\exists y)(\forall x \in a)\, \mathcal{P}(x, y, A, \bar{x}) \\ \Leftrightarrow (\exists y)(\forall^{\mathbf{S}} x \in A)\mathcal{P}(x, y, A, \bar{x})]$$

  where $\mathcal{P}(x, y, A, \bar{x})$ is any statement in the $\in$-language; $\mathcal{P}^{\mathbf{fin}}(A)$ is the set of all finite subsets of A.

Nelson's *Internal Set Theory* **IST** differs from **BST** as follows:

- *Boundedness* is dropped;
- *Bounded Idealization* is replaced by

  Idealization:

$$(\forall \bar{x})[(\forall^{\mathbf{S}} \text{ finite } a)(\exists y)(\forall x \in a)\, \mathcal{P}(x, y, \bar{x}) \Leftrightarrow (\exists y)(\forall^{\mathbf{S}} x)\mathcal{P}(x, y, \bar{x})]$$

  where $\mathcal{P}(x, y, \bar{x})$ is any statement in the $\in$-language.

A thorough discussion of the relative merits of **IST** and **BST** can be found in Kanovei and Reeken's monograph [7]. They also show (Theorem 3.4.5) that the class $\mathbf{B} = \{x \mid (\exists^{\mathbf{S}} y)(x \in y\}$ of *bounded sets* gives an interpretation for **BST** in **IST**. For our purposes, the main advantage of **BST** is that it proves the Reduction Theorem (see Appendix) for *all* statements, not just the bounded ones. This allows us to give a simple formulation of the Stability Principle, which underlies the presentation of analysis in [6].

Relative Bounded Set Theory

Relative Bounded Set Theory **RBST** is formulated in a language with two binary predicate symbols, $\in$ and $\sqsubseteq$. We read $x \sqsubseteq y$ as "*x is observable (or: standard) relative to y.*"

The basic axiom of **RBST** is Relativization.

Relativization:

(1) $(\forall p)(p \sqsubseteq p)$;

(2) $(\forall p)(\forall q)(\forall r)(p \sqsubseteq q \,\wedge\, q \sqsubseteq r \,\rightarrow\, p \sqsubseteq r)$;

(3) $(\forall p)(\forall q)(p \sqsubseteq q \,\vee\, q \sqsubseteq p)$;

(4) $(\forall p)(0 \sqsubseteq p)$;

(5) $(\forall p)(\exists q)(p \sqsubseteq q \,\wedge\, \neg\, q \sqsubseteq p)$.

For the statements of the remaining axioms we use the notation $\mathbf{S}_p(q)$ in place of $q \sqsubseteq p$. Intuitively, $\mathbf{S}_p$ is the universe of objects observable relative to p, and we also write $q \in \mathbf{S}_p$ for $\mathbf{S}_p(q)$.

RBST *postulates that the axioms of* **BST**, *to wit,* **ZFC** *in* $\mathbf{S}$, *Boundedness, Transfer, Standardization and Bounded Idealization, hold with* $\mathbf{S}$ *replaced by* $\mathbf{S}_p$, *for all* p.

Péraire formulated **RIST**, a relativized version of **IST**, in [10]. In Section 4 we use a subtheory of **RIST** denoted there **RIST**$^-$.

RIST$^-$ postulates

- *Relativization*
- *For all* p, **ZFC** *in* **S**, *Transfer, Idealization and Inner Standardization with* **S** *replaced by* $\mathbf{S}_p$.

Inner Standardization is the followig special case of Standardization:

Inner Standardization: $(\forall x)(\exists^{\mathbf{S}} y)(\forall^{\mathbf{S}} z)(z \in y \Leftrightarrow z \in x)$.

It is not clear whether **RIST**$^-$ is truly weaker than **RIST**, but it can be shown that **RBST** proves the bounded analogs of all axioms of **RIST**, such as the multi-level *Idéalisation Contrôlée.*

Structure of the exposition

In Sections 1 and 2 we prove that **IST** is a conservative extensionof **ZFC** by the method from [7], Section 4.4; the original proof in Nelson [9] is different. As mentioned above, **BST** has an interpretation in **IST**, so conservativity of **BST** over **ZFC** follows. This result (and much more) was proved directly in the author's [3].

In Sections 3 and 4 we show that **RIST**$^-$ is a conservative extension of **ZFC**. In the main outline, the proof follows the ideas of the proof by Péraire [10] of an analogous result for **RIST**, but there are many differences in detail.

In Section 5 we prove that the bounded sets of **RIST**$^-$ provide an interpretation of **RBST** in **RIST**$^-$, and thus establish conservativity of **RBST** over **ZFC**. This is also an immediate consequence of the results in [4] and [5], where relative consistency of much stronger theories (**FRBST** and **GRIST**, respectively) has been established by different, necessarily much more complicated, methods.

The Appendix contains the proof of the Reduction Theorem for **BST** and derives the consequences needed in Section 5.

1 Ultrafilters and ultrapowers.

We work in **ZFC** unless stated otherwise. As is customary, we use classes to denote extensions of statements (formulas) of **ZFC**.

Definition 1
*A **filter** over I is a collection F of subsets of I such that*

(1) $\varnothing \notin F; I \in F$;

(2) If $X \in F$ *and* $X \subseteq Y \subseteq I$, *then* $Y \in F$;

(3) If $X, Y \in F$, *then* $X \cap Y \in F$.

*An **ultrafilter** over I is a maximal filter over I (in the ordering of filters by inclusion).*

It is an immediate consequence of Zorn's Lemma that every filter over I can be extended to an ultrafilter over I.

Exercise 1 The following statements are equivalent:

(1) U is an ultrafilter over I;

(2) U is a filter over I with the property:
If $X \cup Y \in U$, then $X \in U$ or $Y \in U$;

(3) U is a filter over I with the property:
For every $X \subseteq I$, either $X \in U$ or $I \setminus X \in U$.

Intuitively, an ultrafilter partitions all subsets of I into two classes: the "large" sets (those in U) and the "small" sets (those not in U).

Example

(1) For a fixed $i \in I$, $U_i = \{X \subseteq I \mid i \in X\}$ *is an ultrafilter over* I*; it is called the* principal ultrafilter generated by i.

(2) Let I be infinite; then $F_\omega = \{X \subseteq I \mid I \setminus X \text{ is finite}\}$ *is a filter; it is called the* free filter *over I.*

Exercise 2 An ultrafilter U over I is nonprincipal if and only if $U \supseteq F_\omega$. Hence over every infinite set I there exist nonprincipal ultrafilters (in fact, there are $2^{2^{|I|}}$ of them).

Definition 2
An ultrafilter U over I is ω-incomplete *if there exists a sequence $(X_n)_{n\in\mathbb{N}}$ such that each $X_n \in U$ and $\bigcap_{n\in\mathbb{N}} X_n = \varnothing$.*

Exercise 3 Every nonprincipal ultrafilter over $\mathbb{N}$ is ω-incomplete.

Let U be a fixed ultrafilter over I. We use it to construct an *interpretation* for the language of **ZFC** (the $\in$-*language*).

Definition 3

$$\mathbb{V}^U = \mathbb{V}^I = \{f : f \text{ is a function, } \operatorname{dom} f = I \text{ and } \operatorname{ran} f \subseteq \mathbb{V}\}.$$

For $f, g \in \mathbb{V}^U$ let

$$f =_U g \text{ iff } \{i \in I : f(i) = g(i)\} \in U;$$

$$f \in_U g \text{ iff } \{i \in I : f(i) \in g(i)\} \in U.$$

*The **ultrapower of** $\mathbb{V}$ **modulo** U is the triple $(\mathbb{V}^U, =_U, \in_U)$.*

If $\mathcal{P}(x_1, \ldots, x_k)$ is any $\in$-statement, we let $\mathcal{P}^U(x_1, \ldots, x_k)$ denote the statement obtained from $\mathcal{P}$ by replacing all occurences of $=$ and $\in$ by $=_U$ and $\in_U$, respectively, and restricting the range of all quantifiers to $\mathbb{V}^U$, [that is, replacing $(\forall x) \ldots$ with $(\forall x)(x \in \mathbb{V}^U \Rightarrow \ldots)$ and $(\exists x) \ldots$ with $(\exists x)(x \in \mathbb{V}^U \wedge \ldots)$; this may involve renaming some bound variables, if necessary or convenient.] We read $\mathcal{P}^U$ as "$\mathcal{P}$ *holds in the ultrapower*."

The ultrapower provides an interpretation of the language of **ZFC**. Intuitively, we think of elements of $\mathbb{V}^U$ as "sets in the sense of the ultrapower," $f =_U g$ means that "f and g are equal in the sense of the ultrapower," and $f \in_U g$ means that "f is an element of g in the sense of the ultrapower."

The fundamental fact about ultrapowers now takes the following form (*Łoś Theorem*):

Theorem 1
Let $\mathcal{P}(x_1, \ldots, x_k)$ be an $\in$-statement. For all $f_1, \ldots, f_k \in \mathbb{V}^U$,

$$\mathcal{P}^U(f_1, \ldots, f_k) \Leftrightarrow \{i \in I : \mathcal{P}(f_1(i), \ldots, f_k(i))\} \in U.$$

Proof: By induction on the complexity of $\mathcal{P}$.

If $\mathcal{P}$ is an atomic statement $x_\ell = x_m$, $(f_\ell = f_m)^U$ is the statement $f_\ell =_U f_m$, which holds if and only if $\{i \in I : f_\ell(i) = f_m(i)\} \in U$. The case of $x_\ell \in x_m$ is similar.

The properites of an ultrafilter easily imply that if the claim is true for $\mathcal{P}_1$ and $\mathcal{P}_2$, then it is also true for $\neg \mathcal{P}_1$ and $\mathcal{P}_1 \wedge \mathcal{P}_2$.

If $\mathcal{P}$ is of the form $(\exists y)\mathcal{Q}(x_1, \ldots, x_k, y)$, $((\exists y)\mathcal{Q}(f_1, \ldots, f_k, y))^U$ is the statement $(\exists g \in \mathbb{V}^U)\mathcal{Q}^U(f_1, \ldots, f_k, g))$. Let $g \in \mathbb{V}^U$ be such that $\mathcal{Q}^U(f_1, \ldots, f_k, g)$. By the inductive assumption, $\{i \in I : \mathcal{Q}(f_1(i), \ldots, f_k(i), g(i))\} \in U$, hence $\{i \in I : (\exists x)\mathcal{Q}(f_1(i), \ldots, f_k(i), x)\} \in U$, i.e., $\{i \in I : \mathcal{P}(f_1(i), \ldots, f_k(i))\} \in U$. Using the Axiom of Choice, the argument can be reversed. $\square$

In particular, $\mathcal{P}^U \Leftrightarrow \mathcal{P}$ holds for any sentence (i.e., a statement with no parameters) $\mathcal{P}$, so all axioms of **ZFC** hold in the ultrapower.

The relation $=_U$ is a congruence with respect to $\in_U$. This suggests the possibility of taking equivalence classes modulo $=_U$, in order to obtain an interpretation where $=$ is interpreted by true equality. To avoid some technical issues, we forego this step and work with elements of $\mathbb{V}_U$ rather than their equivalence classes, similarly as one can work with fractions rather than their equivalence classes (rational numbers).

We next extend this interpretation to the $\in$-**S**-language.

Definition 4
For $x \in \mathbb{V}$, let $\mathfrak{k}_{U,x}$ be the constant function on I with value x; i.e., $\mathfrak{k}_{U,x}(i) = x$ for all $i \in I$. Let

$$\mathbb{S}_U = \{f \in \mathbb{V}^U \mid f =_U \mathfrak{k}_{U,x} \text{ for some } x \in \mathbb{V}\}.$$

Clearly $x = y \Leftrightarrow \mathfrak{k}_{U,x} =_U \mathfrak{k}_{U,y}$ and $x \in y \Leftrightarrow \mathfrak{k}_{U,x} \in_U \mathfrak{k}_{U,y}$, so the mapping $\mathfrak{k}_U : x \mapsto \mathfrak{k}_{U,x}$ is an isomorphism of $(\mathbb{V}, =, \in)$ and $(\mathbb{S}_U, =_U, \in_U)$, in an obvious sense. If $\mathcal{P}$ is a statement in the $\in$-language, $\mathcal{P}^{\mathbb{S}_U}$ denotes the statement obtained from $\mathcal{P}^U$ by restricting all quantifiers to $\mathbb{S}_U$. The above isomorphism implies that

$$(\forall x_1, \ldots, x_k)(\mathcal{P}(x_1, \ldots, x_k) \Leftrightarrow \mathcal{P}^{\mathbb{S}_U}(\mathfrak{k}_{U,x_1}, \ldots, \mathfrak{k}_{U,x_k})).$$

We read $f \in \mathbb{S}_U$ as "f is standard in the sense of the ultrapower." The **extended ultrapower of $\mathbb{V}$ modulo U** is the quadruple $(\mathbb{V}^U, =_U, \in_U, \mathbb{S}_U)$. It is an interpretation for the $\in$-**S**-language. If $\mathcal{P}$ is a statement in this language, we let $\mathcal{P}^U$ be the statement where, in addition, every occurence of $\mathbf{S}(x)$ is replaced by $x \in \mathbb{S}_U$.

Our next goal is to show that all of the axioms of **IST** (see Appendix), except for Idealization, hold in the extended ultrapower of $\mathbb{V}$.

Theorem 2
*Let U be an ultrafilter over I. Then **ZFC** in $\mathbf{S}$, Transfer and Standardization hold in the extended ultrapower of $\mathbb{V}$ modulo U.*

Proof: Let $\mathcal{P}$ be a statement in the $\in$-language. We observe that $(\mathcal{P}^{\mathbf{S}})^U$is equivalent to $\mathcal{P}^{\mathbb{S}_U}$. If $\mathcal{P}$ is an axiom of **ZFC** (written as a sentence, i.e., with no free variables), then $(P^{\mathbf{S}})^U \Leftrightarrow \mathcal{P}^{\mathbb{S}_U} \Leftrightarrow \mathcal{P}$, so $\mathcal{P}$ holds in the standard universe of the extended ultrapower.

To prove that Transfer holds in the extended ultrapower, let $f_1, \ldots, f_k \in \mathbb{S}_U$. Let $f_1 =_U \mathfrak{k}_{U,x_1}, \ldots, f_k =_U \mathfrak{k}_{U,x_k}$. We have

$$(\mathcal{P}^{\mathbf{S}}(f_1, \ldots, f_k))^U \Leftrightarrow (\mathcal{P}^{\mathbf{S}}(\mathfrak{k}_{U,x_1}, \ldots, \mathfrak{k}_{U,x_k}))^U \Leftrightarrow \mathcal{P}^{\mathbb{S}_U}(\mathfrak{k}_{U,x_1}, \ldots, \mathfrak{k}_{U,x_k}) \Leftrightarrow$$

$$\Leftrightarrow \mathcal{P}(x_1, \ldots, x_k) \Leftrightarrow \mathcal{P}^U(\mathfrak{k}_{U,x_1}, \ldots, \mathfrak{k}_{U,x_k}) \Leftrightarrow \mathcal{P}^U(f_1, \ldots, f_k).$$

It remains to prove that Standardization holds in the extended ultrapower. Let $\mathcal{P}(x, y, x_1, \ldots, x_k)$ be a statement in the $\in$-**S**-language. Given $g, f_1, \ldots, f_k \in \mathbb{V}^U$ such that $\operatorname{ran} g \subseteq A$, consider $B = \{x \in A : \mathcal{P}^U(\mathfrak{k}_{U,x}, f_1, \ldots, f_k)\}$. Then $\mathfrak{k}_{U,B} \in \mathbb{S}_U$ and $(\forall f \in \mathbb{S}_U)(f \in_U \mathfrak{k}_{U,B} \Leftrightarrow f \in_U g \wedge \mathcal{P}^U(f, f_1, \ldots, f_k))$. This is exactly what Standardization requires. $\square$

If U is a principal ultrafilter generated by $i \in I$, then $f =_U \mathfrak{k}_{U,f(i)}$ for every $f \in \mathbb{V}^U$, so $\mathbb{S}_U = \mathbb{V}^U$; in the extended ultrapower of $\mathbb{V}$ modulo a principal U there are no nonstandard sets.

Theorem 3
If U is ω-incomplete, then the extended ultrapower of $\mathbb{V}$ modulo U has nonstandard natural numbers.

Proof: Let $(X_n)_{n\in\mathbb{N}}$ be such that each $X_n \in U$ and $\bigcap_{n\in\mathbb{N}} X_n = \varnothing$; without loss of generality we can assume that $X_n \supseteq X_{n+1}$ holds for all n [replace X_n by $\bigcap_{m\le n} X_m$] and $X_0 = I$. Define f on I by: $d(i) = n$ iff $i \in X_n \setminus X_{n+1}$. Then $d \in_U \mathfrak{k}_{U,\mathbb{N}}$ and $d \neq_U \mathfrak{k}_{U,n}$ for any $n \in \mathbb{N}$, because $\{i \in I \mid d(i) = \mathfrak{k}_{U,n}(i)\} = X_n \setminus X_{n+1} \notin U$. □

However, the following exercise shows that the extended ultrapower of $\mathbb{V}$ modulo U never satisfies even Bounded Idealization (see Appendix), and therefore cannot be an interpretation for the full **IST**.

Exercise 4 Let J be an infinite set of cardinality κ. Let $F_\kappa = \{X \subseteq J \mid |J \setminus X| < \kappa\}$. Show that F_κ is a filter over J.
Let $V \supseteq F_\kappa$ be an ultrafilter over J (such ultrafilters are called *uniform*). Show that if $\kappa > |I|$, then there is no $f \in \mathbb{V}^U$ such that $f \in_U \mathfrak{k}_{U,X}$ holds for all $X \in V$. [Hint: Assume to the contrary that such f exists. Since $f \in_U \mathfrak{k}_{U,J}$, the set $\{i \in I \mid f(i) \in J\} \in U$. Let $X = J \cap \operatorname{ran} f$ and show that $X \in V$. This is a contradiction because $|X| \le |I| < \kappa$.]

2 Relative consistency of IST.

In order to obtain an interpretation for full **IST**, we first produce a set-sized interpretation of **ZFC** and then take the ultrapower of this interpretation, modulo a suitable ultrafilter.

We recall von Neumann's cumulative hierarchy of sets $(\mathbb{V}_\alpha)_{\alpha \text{ ordinal}}$:

(1) $\mathbb{V}_0 = \varnothing$;

(2) $\mathbb{V}_{\alpha+1} = \mathcal{P}(\mathbb{V}_\alpha)$;

(3) $\mathbb{V}_\lambda = \bigcup_{\alpha<\lambda} \mathbb{V}_\alpha$ for $\lambda > 0$ limit.

ZFC proves that $\mathbb{V} = \bigcup_{\alpha \text{ ordinal}} \mathbb{V}_\alpha$.

The theory **ZFC**$\boldsymbol{\vartheta}$ (see [7]) is formulated in the $\in$-$\boldsymbol{\vartheta}$-language, where $\boldsymbol{\vartheta}$ is a constant symbol. Its axioms are

(1) All the axioms of **ZFC**, with the understanding that the symbol $\boldsymbol{\vartheta}$ can appear in the axioms of Separation and Replacement.

(2) $\boldsymbol{\vartheta} > 0$ is a limit ordinal.

(3) For every statement $\mathcal{P}(x_1, \ldots, x_k)$ in the $\in$-language ($\boldsymbol{\vartheta}$ not allowed!),

$$(\forall x_1, \ldots, x_k \in \mathbb{V}_{\boldsymbol{\vartheta}})(\mathcal{P}(x_1, \ldots, x_k) \Leftrightarrow \mathcal{P}_{\boldsymbol{\vartheta}}(x_1, \ldots, x_k)),$$

where $\mathcal{P}_{\boldsymbol{\vartheta}}$ is obtained from $\mathcal{P}$ by restricting all quantifiers to $\mathbb{V}_{\boldsymbol{\vartheta}}$.

A consequence of the last item is that $\mathcal{P}_{\boldsymbol{\vartheta}}$ holds for every axiom $\mathcal{P}$ of **ZFC**; in other words, $(\mathbb{V}_{\boldsymbol{\vartheta}}, =, \in)$ is an interpretation of **ZFC** in **ZFC**$\boldsymbol{\vartheta}$ [it is understood that $=$ and $\in$ stand here for the restrictions of these relations to $\mathbb{V}_{\boldsymbol{\vartheta}}$]. For our purposes it is important that the universe $\mathbb{V}_{\boldsymbol{\vartheta}}$ of this interpretation is a *set*.

Note: One cannot prove in **ZFC**$\boldsymbol{\vartheta}$ that $(\mathbb{V}_{\boldsymbol{\vartheta}}, =, \in)$ is a *model* of **ZFC** in the sense of model theory. This would prove consistency of **ZFC** in **ZFC**$\boldsymbol{\vartheta}$ and, in conjunction with the next theorem, contradict Gödel's Second Incompleteness Theorem.

Theorem 4
*If **ZFC** is consistent, then **ZFC**$\boldsymbol{\vartheta}$ is consistent.*

Proof: If **ZFC**$\boldsymbol{\vartheta}$ proved a contradiction, the proof would use only a finite list of instances of axioms from group (3); say for the statements $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$. But the Reflection Principle of **ZFC** (see e.g. Kunen [8]) implies that there is a limit ordinal $\theta > 0$ such that (1), (2) and the instances of (3) for $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ hold for this θ. Hence a contradiction could be proved in **ZFC**. □

We now work in **ZFC**$\boldsymbol{\vartheta}$ and carry out the construction of the extended ultrapower from Section 1, but with the proper class $\mathbb{V}$ replaced by the *set* $\mathbb{V}_{\boldsymbol{\vartheta}}$.

Thus the sets in the sense of the interpretation are functions with domain I and values in $\mathbb{V}_{\boldsymbol{\vartheta}}$. The relations $=_U, \in_U$ and $\mathbb{S}_U$ are now restrictions to $\mathbb{V}_{\boldsymbol{\vartheta}}^U = \mathbb{V}_{\boldsymbol{\vartheta}}^I$

of the corresponding relations from Section 1. The **extended ultrapower of $\mathbb{V}_{\boldsymbol{\vartheta}}$ modulo U** is the quadruple $(\mathbb{V}_{\boldsymbol{\vartheta}}^U, =_U, \in_U, \mathbb{S}_U)$.

Theorem 5
ZFC *in* **S**, *Transfer and Standardization hold in the extended ultrapower of* $\mathbb{V}_{\boldsymbol{\vartheta}}$ *modulo* U.

Proof: Repeat the arguments in Section 1 with $\mathbb{V}_{\boldsymbol{\vartheta}}$ in the place of $\mathbb{V}$. □

We complete the construction of an interpretation for **IST** by showing that Idealization holds in the extended ultrapower of $\mathbb{V}_{\boldsymbol{\vartheta}}$ modulo U for a suitable choice of the ultrafilter U.

Let $\mathcal{P}^{\text{fin}}(I)$ denote the collection of all finite subsets of I.

Definition 5
Let B, C be functions defined on $\mathcal{P}^{\text{fin}}(I)$ and with values in the ultrafilter U.
The function B is monotone *if $a \subseteq b$ implies $B(a) \supseteq B(b)$, for all $a, b \in \mathcal{P}^{\text{fin}}(I)$.*
The function B is additive *if $B(a \cup b) = B(a) \cap B(b)$, for all $a, b \in \mathcal{P}^{\text{fin}}(I)$.*
We say that C is subordinate *to B if $C(a) \subseteq B(a)$, for all $a \in \mathcal{P}^{\text{fin}}(I)$.*

Definition 6
An ultrafilter U over I is good *if it is ω-incomplete and for every monotone function B there is an additive function C subordinate to B.*

We refer to Chang and Keisler [2] for a proof that for every infinite set I there exist good (κ^+-good, for $\kappa = |I|$) ultrafilters over I.

Theorem 6
Let U be a good ultrafilter over $I = \mathbb{V}_{\boldsymbol{\vartheta}}$. Then ***IST*** *holds in the extended ultrapower of $\mathbb{V}_{\boldsymbol{\vartheta}}$ modulo U.*

Proof: It remains to prove that Idealization holds. Let $\mathcal{P}(x_1, \ldots, x_k)$ be a statement in the $\in$-language. Let $h_1, \ldots, h_k \in \mathbb{V}_{\boldsymbol{\vartheta}}^U$.

Assume that for every finite set $a \in \mathbb{V}_{\boldsymbol{\vartheta}}$

$$(\exists g \in \mathbb{V}_{\boldsymbol{\vartheta}}^U)(\forall f \in_U \mathfrak{k}_a)\mathcal{P}_{\boldsymbol{\vartheta}}^U(f, g, h_1, \ldots, h_k).$$

Let $D(a) = \{i \in I \mid (\exists y \in \mathbb{V}_{\boldsymbol{\vartheta}})(\forall x \in a)\mathcal{P}_{\boldsymbol{\vartheta}}(x, y, h_1(i), \ldots, h_k(i))\}$. By Łoś Theorem, $D(a) \in U$.

Let $(I_n)_{n\in\mathbb{N}}$ be such that $I_0 = I$, $I_{n+1} \subseteq I_n$ and $I_n \in U$, for all $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} I_n = \varnothing$. We define $B(a) = D(a) \cap I_n$ where $n = |a|$, and notice that $B(a) \in U$ and the function $(B(a) \mid a \in \mathcal{P}^{\text{fin}}(\mathbb{V}_{\boldsymbol{\vartheta}}))$ is monotone. Let $(C(a) \mid a \in \mathcal{P}^{\text{fin}}(\mathbb{V}_{\boldsymbol{\vartheta}}))$ be an additiva function subordinate to B. For each $i \in I$ let $a_i = \{x \in \mathbb{V}_{\boldsymbol{\vartheta}} \mid i \in C(\{x\})\}$. The set a_i is finite, because existence of an infinite sequence $(x_n)_{n\in\mathbb{N}}$ of distinct elements of a_i would imply $i \in \bigcap_{j\leq n} C(\{x_j\}) = C(\{x_0, \ldots, x_n\}) \subseteq I_n$, contradicting $\bigcap_{n\in\mathbb{N}} I_n = \varnothing$.

For every $i \in I$ we have $i \in C(\{x\})$ for all $x \in a_i$. So $i \in \bigcap_{x\in a_i} C(\{x\}) = C(a_i) \subseteq B(a_i)$ and we can choose $f(i) \in \mathbb{V}_{\boldsymbol{\vartheta}}$ such that

$$(\forall x \in a_i)\mathcal{P}_{\boldsymbol{\vartheta}}(x, f(i), h_1(i), \ldots, h_k(i)).$$

We claim that $(\forall x \in \mathbb{V}_{\boldsymbol{\vartheta}})\mathcal{P}^U_{\boldsymbol{\vartheta}}(\mathfrak{k}_x, f, h_1, \ldots, h_k)$.

Fix $x \in \mathbb{V}_{\boldsymbol{\vartheta}}$. By definition, $\{i \in I \mid x \in a_i\} \subseteq \{i \in I \mid \mathcal{P}_{\boldsymbol{\vartheta}}(x, f(i), h_1(i), \ldots, h_k(i))\}$. But $\{i \in I \mid x \in a_i\} = \{i \in I \mid i \in C(\{x\})\} = C(\{x\}) \in U$, so

$$\{i \in I \mid \mathcal{P}_{\boldsymbol{\vartheta}}(x, f(i), h_1(i), \ldots, h_k(i))\} \in U$$

and the conclusion follows by Łoś Theorem.

The opposite implication is trivial, because one can prove in **IST**, without appeal to Idealization, that all elements of a standard finite set are standard. □

Theorem 7
IST *is conservative over* ***ZFC****. This means that, for any statement $\mathcal{P}$ in the $\in$-language, if* ***IST*** *proves $\mathcal{P}^{\mathbf{S}}$, then* ***ZFC*** *proves $\mathcal{P}$.*
In particular, if ***ZFC*** *is consistent, then* ***IST*** *is consistent.*

Proof: Suppose that **IST** proves $\mathcal{P}^{\mathbf{S}}$ but **ZFC** does not prove $\mathcal{P}$. Then the theory $\mathbf{ZFC}^+ = \mathbf{ZFC} + \neg\mathcal{P}$ is consistent. Let $\mathbf{ZFC}^+_{\boldsymbol{\vartheta}}$ be the theory obtained by adding $\neg\mathcal{P}$ to the axioms of $\mathbf{ZFC}_{\boldsymbol{\vartheta}}$. The proof of Theorem 4 goes through to show that $\mathbf{ZFC}^+_{\boldsymbol{\vartheta}}$ is consistent; in this theory $\neg\mathcal{P}_{\boldsymbol{\vartheta}}$ holds. Then $\neg\mathcal{P}^{\mathbf{S}}$ holds in the extended ultrapower of $\mathbb{V}_{\boldsymbol{\vartheta}}$ modulo a good ultrafilter U. But this extended ultrapower satisfies **IST**, hence in particular its consequence $\mathcal{P}^{\mathbf{S}}$. This is a contradiction.

Suppose **IST** proved a contradiction. Then it would prove every statement, for example, $(\exists x)(x \neq x)$. By conservativity, **ZFC** would then also prove $(\exists x)(x \neq x)$. □

3 Repeated ultrapowers.

In this section we carry out a preliminary technical step for the construction of an interpretation for **RIST**$^-$: Given an interpretation $(\mathbb{V}^U_{\boldsymbol{\vartheta}}, =_U, \in_U, \widetilde{\mathbb{S}})$ where **IST** holds [this is true for $\widetilde{\mathbb{S}} = \mathbb{S}_U$, but we need a more general result], we construct a new interpretation for **IST** by taking its ultrapower, and describe this ultrapower explicitly.

Let U be an ultrafilter over I and V an ultrafilter over J.

Definition 7

$$(\mathbb{V}^U_{\boldsymbol{\vartheta}})^V = \{f : \operatorname{dom} f = J \text{ and } \operatorname{ran} f \subseteq \mathbb{V}^U_{\boldsymbol{\vartheta}}\};$$

For $f, g \in (\mathbb{V}^U_{\boldsymbol{\vartheta}})^V$*:*

$$f =_{U,V} g \Leftrightarrow \{j \in J \mid f(j) =_U g(j)\} \in V;$$

$$f \in_{U,V} g \Leftrightarrow \{j \in J \mid f(j) \in_U g(j)\} \in V;$$

$$f \in \widetilde{\mathbb{S}}_V \Leftrightarrow \{j \in J \mid f(j) \in \widetilde{\mathbb{S}}\} \in V.$$

It is an easy exercise to prove Łoś Theorem for this ultrapower.

Theorem 8
Let $\mathcal{P}(x_1, \ldots, x_k)$ *be a statement in the* $\in$-**S**-*language. Let* $\mathcal{P}^{U,V}_{\boldsymbol{\vartheta}}$ *be obtained from* $\mathcal{P}$ *by replacing each occurence of* $=, \in, \mathbf{S}$ *by* $=_{U,V}, \in_{U,V}, \widetilde{\mathbb{S}}_V$*, respectively, and restricting all quantifiers to* $(\mathbb{V}^U_{\boldsymbol{\vartheta}})^V$*.*
For all $f_1, \ldots, f_k \in \mathbb{V}^U_{\boldsymbol{\vartheta}}$*,*

$$\mathcal{P}^{U,V}_{\boldsymbol{\vartheta}}(f_1, \ldots, f_k) \Leftrightarrow \{j \in J \mid \mathcal{P}^U_{\boldsymbol{\vartheta}}(f_1(j), \ldots, f_k(j))\} \in V.$$

Corollary *If **IST** holds in the interpretation* $(\mathbb{V}^U_{\boldsymbol{\vartheta}}, =_U, \in_U, \widetilde{\mathbb{S}})$*, then **IST** holds also in the interpretation* $((\mathbb{V}^U_{\boldsymbol{\vartheta}})^V, =_{U,V}, \in_{U,V}, \widetilde{\mathbb{S}}_V)$*.*

Let now $\mathcal{P}$ be an $\in$-statement. Then one can use Łoś Theorem for the ultrapower modulo U to write further:

$$\mathcal{P}^{U,V}_{\boldsymbol{\vartheta}}(f_1, \ldots, f_k) \Leftrightarrow \{j \in J \mid \{i \in I \mid \mathcal{P}_{\boldsymbol{\vartheta}}(f_1(j)(i), \ldots, f_k(j)(i))\} \in U\} \in V.$$

This equivalence suggests the following definition.

Definition 8
For $Z \subseteq J \times I$ *let*

$$Z \in V \otimes U \Leftrightarrow \{j \in J \mid \{i \in I \mid \langle j, i\rangle \in Z\} \in U\} \in V.$$

Exercise 5 $V \otimes U$ is an ultrafilter over $J \times I$.
For $X \subseteq I$, $(X \times I) \in V \otimes U \Leftrightarrow X \in V$.

For every $f \in (\mathbb{V}_{\boldsymbol{\vartheta}}^U)^V$ define $\widehat{f} \in \mathbb{V}_{\boldsymbol{\vartheta}}^{V\otimes U}$ by $\widehat{f}(\langle j,i\rangle) = f(j)(i)$. The mapping $\widehat{\ }$ is onto $\mathbb{V}_{\boldsymbol{\vartheta}}^{V\otimes U}$ and preserves $=$ and $\in$ in the following sense:

$$f =_{U,V} g \Leftrightarrow \widehat{f} =_{V\otimes U} \widehat{g}; \qquad f \in_{U,V} g \Leftrightarrow \widehat{f} \in_{V\otimes U} \widehat{g}.$$

Moreover,

$$f \in (\mathbb{S}_U)_V \Leftrightarrow \{j \in J \mid f(j) \in \mathbb{S}_U\} \in V \Leftrightarrow (\exists g \in \mathbb{V}_{\boldsymbol{\vartheta}}^V)(\{j \in J \mid f(j) =_U \mathfrak{k}_{U,g(j)}\} \in V) \Leftrightarrow$$

$$\Leftrightarrow (\exists g \in \mathbb{V}_{\boldsymbol{\vartheta}}^V)(f =_{V\otimes U} \mathfrak{k}_{V,U,g}),$$

where $\mathfrak{k}_{V,U,g} \in \mathbb{V}^{V\otimes U}$ is defined by $\mathfrak{k}_{V,U,g}(\langle j,i\rangle) = g(j)(i)$.

We let

$$\mathbb{S}_{V,V\otimes U} = \{h \in \mathbb{V}^{V\otimes U} \mid (\exists g \in \mathbb{V}_{\boldsymbol{\vartheta}}^V)(\{\langle j,i\rangle \in J \times I \mid h(\langle j,i\rangle) = g(j)\} \in V \otimes U\}).$$

We summarize these observations.

Theorem 9
The mapping $\widehat{\ }$ *is an isomorphism of the interpretations* $((\mathbb{V}_{\boldsymbol{\vartheta}}^U)^V, =_{U,V}, \in_{U,V}, (\mathbb{S}_U)_V)$ *and* $(\mathbb{V}_{\boldsymbol{\vartheta}}^{V\otimes U}, =_{V\otimes U}, \in_{V\otimes U}, \mathbb{S}_{V,V\otimes U})$.

Corollary *If U is a good ultrafilter over $I = \mathbb{V}_{\boldsymbol{\vartheta}}$, then **IST** holds in the interpretation* $(\mathbb{V}_{\boldsymbol{\vartheta}}^{V\otimes U}, =_{V\otimes U}, \in_{V\otimes U}, \mathbb{S}_{V,V\otimes U})$.

For the construction in the next section we need a generalization of Theorem 6.

Theorem 10
*If U is a good ultrafilter over $I = \mathbb{V}_{\boldsymbol{\vartheta}}$, then **IST** holds in the interpretation* $(\mathbb{V}_{\boldsymbol{\vartheta}}^{U\otimes V}, =_{U\otimes V}, \in_{U\otimes V}, \mathbb{S}_{U\otimes V})$.

Proof: All axioms of **IST** except Idealization hold on account of Theorem 2.

As in the discussion preceding Theorem 9, but with the roles of U and V exchanged, we establish an isomorphism $\widehat{\ }$ of the interpretation $((\mathbb{V}_{\boldsymbol{\vartheta}}^V)^U, =_{V,U}, \in_{V,U}, \mathbb{S})$ and the interpretation $(\mathbb{V}_{\boldsymbol{\vartheta}}^{U\otimes V}, =_{U\otimes V}, \in_{U\otimes V}, \mathbb{S}_{U\otimes V})$; now $\widehat{f}(\langle i,j\rangle) = f(i)(j)$. Here, by definition, $f \in \mathbb{S}$ iff there exists $x \in \mathbb{V}_{\boldsymbol{\vartheta}}$ such that $\{i \in I \mid f(i) =_V \mathfrak{k}_{V,x}\} \in U$, i.e., $\{\langle i,j\rangle \in I \times J \mid \widehat{f}(i,j) = x\} \in U \otimes V$; so we have $f \in \mathbb{S} \Leftrightarrow \widehat{f} \in \mathbb{S}_{U\otimes V}$.

It now suffices to prove that Idealization holds in $((\mathbb{V}_{\boldsymbol{\vartheta}}^V)^U, =_{V,U}, \in_{V,U}, \mathbb{S})$. To do that, we repeat the argument from the proof of Theorem 6, with $\mathbb{V}_{\boldsymbol{\vartheta}}$ replaced by $\mathbb{V}_{\boldsymbol{\vartheta}}^V$. We indicate the main changes.

The parameters $h_1, \ldots, h_k$ are now in $(\mathbb{V}_{\boldsymbol{\vartheta}}^V)^U$. We assume that for every finite set $a \in \mathbb{V}_{\boldsymbol{\vartheta}}$

$$(\exists f \in (\mathbb{V}_{\boldsymbol{\vartheta}}^V)^U)(\forall x \in a)\mathcal{P}_{\boldsymbol{\vartheta}}^{V,U}(\mathfrak{k}_{V,U,x}, f, h_1, \ldots, h_k)$$

and let $E_a = \{i \in I \mid (\exists y \in \mathbb{V}_{\boldsymbol{\vartheta}}^V)(\forall x \in a)\mathcal{P}_{\boldsymbol{\vartheta}}^V(\mathfrak{k}_{V,x}, y, h_1(i), \ldots, h_k(i))\}$. The argument produces a function $f \in (\mathbb{V}_{\boldsymbol{\vartheta}}^V)^U$ such that for every $x \in \mathbb{V}_{\boldsymbol{\vartheta}}$

$$\{i \in I \mid \mathcal{P}_{\boldsymbol{\vartheta}}^V(\mathfrak{k}_{V,x}, f(i), h_1(i), \ldots, h_k(i))\} \in U,$$

and the conclusion $(\forall x \in \mathbb{V}_{\boldsymbol{\vartheta}})\mathcal{P}_{\boldsymbol{\vartheta}}^{V,U}(\mathfrak{k}_{V,U,x}, f, h_1, \ldots, h_k)$ follows by Łoś Theorem. $\square$

4 Relative consistency of RIST$^-$.

Throughout this section we work in **ZFC$\boldsymbol{\vartheta}$** and assume that U is a good ultrafilter over $I = \mathbb{V}_{\boldsymbol{\vartheta}}$.

Definition 9
Define by recursion:

(1) $U_0 = \{\{\varnothing\}\}$*; this is the principal ultrafilter over* $I^0 = \{\varnothing\}$*;*

(2) $U_1 = U$*;* $I^1 = I$*;*

(3) $U_{n+1} = U_n \otimes U$*; this is an ultrafilter over* $I^n \times I = I^{n+1}$*.*

Exercise 6 If $n = k + \ell$, then $U_n = U_k \otimes U_\ell$.
(Actually, this is true only up to the isomorphism that identifies I^n with $I^k \times I^\ell$, but we ignore this pedantic distinction.)

Let $\mathbb{V}^n_{\boldsymbol{\vartheta}} = \mathbb{V}^{U_n}_{\boldsymbol{\vartheta}}$, and for $f, g \in \mathbb{V}^n_{\boldsymbol{\vartheta}}$ let $f =_n g$ iff $f =_{U_n} g$, $f \in_n g$ iff $f \in_{U_n} g$.
For $k < n$ let $\mathbb{S}_{k,n} = \mathbb{S}_{U_k,U_n} = \{f \in \mathbb{V}^n_{\boldsymbol{\vartheta}} \mid (\exists g \in \mathbb{V}^k_{\boldsymbol{\vartheta}})(f =_n \mathfrak{k}_{U_k,U_{n-k},g})\}$.

Theorem 11
IST *holds in the interpretation* $(\mathbb{V}^n_{\boldsymbol{\vartheta}}, =_n, \in_n, \mathbb{S}_{k,n})$*, for each* $n > k$*.*

Proof: For $k = 0$ this is just the interpretation $(\mathbb{V}^{U\otimes V}, =_{U\otimes V}, \in_{U\otimes V}, \mathbb{S}_{U\otimes V})$ from Theorem 10, where one lets $V = U_{n-1}$, so that $U \otimes V = U_n$.

By the Corollary to Theorem 8, **IST** holds also in the interpretation $((\mathbb{V}^{U^n}_{\boldsymbol{\vartheta}})^V, =_{U^n,V}, \in_{U^n,V}, (\mathbb{S}_{0,n})_V)$. For $k > 0$ we now take $V = U^k$ and define the mapping $\widehat{\ }$ by

$$\widehat{f}(\langle i_0, \ldots, i_{k-1}, i_k, \ldots, i_{k+n-1}\rangle = f(i_0, \ldots, i_{k-1})(i_k, \ldots, i_{k+n-1}),$$

for $f \in (\mathbb{V}^{U^n}_{\boldsymbol{\vartheta}})^V$. As in the discussion preceding Theorem 9, one shows easily that $\widehat{\ }$ is an isomorphism of this interpretation and the interpretation $(\mathbb{V}^{k+n}_{\boldsymbol{\vartheta}}, =_{k+n}, \in_{k+n}, \mathbb{S}_{k,k+n})$, which therefore also satisfies **IST**. $\square$

Definition 10
For $n \le m \in \mathbb{N}$ *and* $f \in \mathbb{V}^n_{\boldsymbol{\vartheta}}$ *let* $\mathfrak{i}_{n,m}(f) \in \mathbb{V}^m_{\boldsymbol{\vartheta}}$ *be defined by* $\mathfrak{i}_{n,m}(f)(\langle x_0, \ldots, x_{m-1}\rangle) = f(\langle x_0, \ldots, x_{n-1}\rangle)$*.*
[For $n = 0$*, by definition* $\langle x_0, \ldots, x_{n-1}\rangle = \varnothing$*.]*

We finally let $\mathbb{V}^*_{\boldsymbol{\vartheta}} = \bigcup_{n\in\mathbb{N}} \mathbb{V}^n_{\boldsymbol{\vartheta}}$, and for $f, g \in \mathbb{V}^*_{\boldsymbol{\vartheta}}$ let
$f =^* g$ iff $f \in \mathbb{V}^n_{\boldsymbol{\vartheta}}$, $g \in \mathbb{V}^m_{\boldsymbol{\vartheta}}$ and $\mathfrak{i}_{n,k}(f) =_k \mathfrak{i}_{m,k}(g)$ for $k = \max\{n, m\}$;
$f \in^* g$ iff $f \in \mathbb{V}^n_{\boldsymbol{\vartheta}}$, $g \in \mathbb{V}^m_{\boldsymbol{\vartheta}}$ and $\mathfrak{i}_{n,k}(f) \in_k \mathfrak{i}_{m,k}(g)$ for $k = \max\{n, m\}$;
$f \in \mathbb{S}^*_n$ iff $(\exists g \in \mathbb{V}^n_{\boldsymbol{\vartheta}})(f =^* g)$.

Exercise 7 Prove that $f =^* g$ iff $f \in \mathbb{V}^n_{\boldsymbol{\vartheta}}$, $g \in \mathbb{V}^m_{\boldsymbol{\vartheta}}$ and $\mathfrak{i}_{n,k}(f) =_k \mathfrak{i}_{m,k}(g)$ holds for all $k \ge \max\{n, m\}$; similarly for $f \in^* g$.

We note that $\mathbb{V}^n_{\boldsymbol{\vartheta}} \subseteq \mathbb{S}^*_n$, and conversely, for every $f \in \mathbb{S}^*_n$ there exists $g \in \mathbb{V}^n_{\boldsymbol{\vartheta}}$ such that $f =^* g$. It is easy to prove from this that

IST *holds in the interpretation* $(\mathbb{S}^*_n, =^*, \in^*, \mathbb{S}^*_k)$*, for every* $k < n$.

We leave this as an exercise. [The identity mapping $Id : f \mapsto f$ is essentially an isomorphism of the interpretation $(\mathbb{V}^n_{\boldsymbol{\vartheta}}, =_n, \in_n, \mathbb{S}_{k,n})$, where **IST** holds, and the interpretation $(\mathbb{S}^*_n, =^*, \in^*, \mathbb{S}^*_k)$, if an allowance is made for equality in both interpretations being only a congruence. (By taking equivalence classes modulo $=_n$ and $=^*$, respectively, one can convert Id into a genuine isomorphism, but it seems simpler to argue directly.)]

Definition 11
For $f \in \mathbb{V}^*_{\boldsymbol{\vartheta}}$*, let* $n(f)$ *be the least* $n \in \mathbb{N}$ *for which* $f \in \mathbb{S}^*_n$.
For $f, g \in \mathbb{V}^*_{\boldsymbol{\vartheta}}$ *we define:* $f \sqsubseteq^* g$ *iff* $n(f) \le n(g)$.

The quadruple $(\mathbb{V}^*_{\boldsymbol{\vartheta}}, =^*, \in^*, \sqsubseteq^*)$ is an interpretation for the $\in$-$\sqsubseteq$-language. For any statement $\mathcal{P}$ in this language, $\mathcal{P}^*$ is the statement obtained from $\mathcal{P}$ by replacing all occurrences of $=$, $\in$ and $\sqsubseteq$ by $=^*$, $\in^*$ and $\sqsubseteq^*$, respectively, and restricting all quantifiers to $\mathbb{V}^*_{\boldsymbol{\vartheta}}$. See the Appendix for the axioms of **RIST**$^-$.

Theorem 12
RIST$^-$ *holds in the interpretation* $(\mathbb{V}^*_{\boldsymbol{\vartheta}}, =^*, \in^*, \sqsubseteq^*)$.

Proof: *Relativization* is trivial and left as an exercise [for (4), note that 0 is interpreted by the constant function h on I^0 with value 0, and $n(h) = 0$].

Note also that $\{g \in \mathbb{V}^*_{\boldsymbol{\vartheta}} \mid g \sqsubseteq^* f\} = \{g \in \mathbb{V}^*_{\boldsymbol{\vartheta}} \mid n(g) \le n(f)\} = \mathbb{S}^*_{n(f)}$. Thus the universes $\mathbf{S}_f$ of **RIST**$^-$ are interpreted as $\mathbb{S}^*_{n(f)}$. It remains only to prove that **ZFC** in **S**, Transfer, Idealization and Inner Standardization hold in the interpretation $(\mathbb{V}^*_{\boldsymbol{\vartheta}}, =^*, \in^*, \mathbb{S}^*_n)$, for every $n \in \mathbb{N}$.

ZFC *in* **S***:*
If $\mathcal{P}$ is a axiom of **ZFC**, then $\mathcal{P}^{\mathbb{S}^*_n}$ holds because $\mathcal{P}$ holds in $(\mathbb{S}^*_n, =^*, \in^*)$.

Transfer:
Let $\mathcal{P}$ be a statement in the $\in$-language. We begin by observing that

$$(\forall n < m)(\forall f_1, \ldots, f_k \in \mathbb{S}^*_n)(\mathcal{P}^{\mathbb{S}^*_n}(f_1, \ldots, f_k) \Leftrightarrow \mathcal{P}^{\mathbb{S}^*_m}(f_1, \ldots, f_k));$$

this is just Transfer in the interpretation $(\mathbb{S}^*_m, =^*, \in^*, \mathbb{S}^*_n)$. We prove by induction on the complexity of statements:

$$(\forall n)(\forall f_1, \ldots, f_k \in \mathbb{S}^*_n)(\mathcal{P}^{\mathbb{S}^*_n}(f_1, \ldots, f_k) \Leftrightarrow \mathcal{P}^*(f_1, \ldots, f_k)).$$

For atomic statements of the form $f_1 = f_2$ and $f_1 \in f_2$ this is trivial. If the claim is true for $\mathcal{P}_1$ and $\mathcal{P}_2$, then, also trivially, it is true for $\mathcal{P}_1 \wedge \mathcal{P}_2$ and $\neg\,\mathcal{P}_1$.

So consider $\mathcal{P}$ of the form $(\exists g)\mathcal{Q}(g, f_1, \ldots, f_k)$. If $(\exists g \in \mathbb{S}^*_n)\mathcal{Q}^{\mathbb{S}^*_n}(g, f_1, \ldots, f_k)$, fix such g. Then $\mathcal{Q}^{\mathbb{S}^*_n}(g, f_1, \ldots, f_k)$ and, by the inductive assumption, $\mathcal{Q}^*(g, f_1, \ldots, f_k)$; hence also $(\exists g \in \mathbb{V}^*_{\boldsymbol{\vartheta}})\mathcal{Q}^*(g, f_1, \ldots, f_k)$.

Conversely, suppose that $(\exists g \in \mathbb{V}^*_{\boldsymbol{\vartheta}})\mathcal{Q}^*(g, f_1, \ldots, f_k)$. Fix such g; wlog. $g \in \mathbb{S}^*_m$ for $m > n$. By the inductive assumption, $\mathcal{Q}^{\mathbb{S}^*_m}(g, f_1, \ldots, f_k)$, hence $\mathcal{P}^{\mathbb{S}^*_m}(f_1, \ldots, f_k)$ and $\mathcal{P}^{\mathbb{S}^*_n}(f_1, \ldots, f_k)$, by the above observation.

Idealization:

Let $\mathcal{P}(x, y, x_1, \ldots, x_k)$ be an $\in$-statement.

Fix $m > n$ so that $h_1, \ldots, h_k \in \mathbb{S}^*_m$; we write $\overline{h}$ for $h_1, \ldots, h_k$. Suppose that

$$(\forall \text{ finite } a)(\exists x)(\forall y \in a)\mathcal{P}(x, y, \overline{h})$$

holds in $(\mathbb{V}^*_{\boldsymbol{\vartheta}}, =^*, \in^*, \mathbb{S}^*_n)$ and let $a \in \mathbb{S}^*_n$ be finite in the sense of the interpretation. The statement $(\exists x)(\forall y \in a)\mathcal{P}(x, y, A, \overline{h})$ has parameters in $\mathbb{S}^*_m$, so by Transfer it holds in $\mathbb{S}^*_m$. But Idealization holds in $(\mathbb{S}^*_m, =^*, \in^*, \mathbb{S}^*_n)$, so there exists $x \in \mathbb{S}^*_m$ such that for every $y \in \mathbb{S}^*_n$ we have $\mathcal{P}^{\mathbb{S}^*_m}(x, y, \overline{h})$ and hence also $\mathcal{P}^*(x, y, \overline{h})$. This establishes the conclusion of Idealization. The other direction is trivial.

Inner Standardization: Let $f \in \mathbb{V}^*_{\boldsymbol{\vartheta}}$ and take $m > n$ such that $f \in \mathbb{S}^*_m$. By Standardization in $(\mathbb{S}^*_m, =^*, \in^*, \mathbb{S}^*_n)$ there is $g \in \mathbb{S}^*_n$ such that for all $h \in \mathbb{S}^*_n$ we have $h \in^* g \Leftrightarrow h \in^* f$. This is precisely what Inner Standardization in the interpretation $(\mathbb{V}^*_{\boldsymbol{\vartheta}}, =^*, \in^*, \mathbb{S}^*_n)$ requires.

$\square$

Corollary

RIST$^-$ *is a conservative extension of* **ZFC**.

Proof: Argue as in the proof of Theorem 7. $\square$

5 RBST

Here we finally reach the objective of these notes and establish conservativity of **RBST** over **ZFC**. To this purpose we obtain an interpretation of **RBST** in **RIST**$^-$.

In this section we work in **RIST**$^-$. We recall that $\mathbf{S}_p = \{x \mid x \sqsubseteq p\}$; if $p \sqsubseteq q$, then $\mathbf{S}_p \subseteq \mathbf{S}_q$. The universe of standard sets $\mathbf{S}_0 = \{x \mid x \sqsubseteq 0\} = \bigcap_p \mathbf{S}_p$. We also write $\mathbf{S}_\infty = \{x \mid x = x\} = \bigcup_p \mathbf{S}_p$ for the universe of all sets.

Let $\mathcal{P}(\bar{x})$ be an $\in$-statement ($\bar{x}$ is shorthand for a list $x_1, \ldots, x_k$). We write $\mathcal{P}_p$ for $\mathcal{P}^{\mathbf{S}_p}$; of course, $\mathcal{P}_\infty$ is (equivalent to) just $\mathcal{P}$. The Transfer Principle in **RIST**$^-$ implies:

$$\text{For all } p \sqsubseteq q \text{ and all } \bar{x} \in \mathbf{S}_p : \quad \mathcal{P}_p(\bar{x}) \Leftrightarrow \mathcal{P}(\bar{x}) \Leftrightarrow \mathcal{P}_q(\bar{x}).$$

Definition 12
$\mathbf{B} = \{x \mid (\exists y \in \mathbf{S}_0)(x \in y)\}$. *If* $x \in \mathbf{B}$, *we say that* x *is* bounded.

Theorem 13
$x \in \mathbf{B}$ *iff* $(\exists y \in \mathbf{S}_0)(x \subseteq y)$.

Proof: If $x \in y$ for $y \in \mathbb{S}_0$, then $x \subseteq \cup y$ and $\cup y \in \mathbb{S}_0$.
If $x \subseteq y$ and $y \in \mathbb{S}_0$, then $x \in \mathcal{P}(y)$ (the power set of y) and $\mathcal{P}(y) \in \mathbb{S}_0$. $\square$

We consider the interpretation $(\mathbf{B}, =, \in, \sqsubseteq)$ of the $\in$-$\sqsubseteq$-language in **RIST**$^-$ (it is understood that the relations are restricted to $\mathbf{B}$).

Let $\mathbf{S}^b_p = \mathbf{S}_p \cap \mathbf{B}$; note that $\mathbf{S}^b_0 = \mathbf{S}_0$ and $\mathbf{S}^b_\infty = \mathbf{B}$.

If $\mathcal{P}$ is an $\in$-statement, we write $\mathcal{P}^b_p$ for $\mathcal{P}^{\mathbf{S}^b_p}$; $\mathcal{P}^b$ is $\mathcal{P}^b_\infty$, i.e., $\mathcal{P}^{\mathbf{B}}$.

Theorem 14 (p is a set or $p = \infty$)

$$(\forall \bar{x} \in \mathbf{S}^b_p)[(\exists y \in \mathbf{S}_p)\mathcal{P}_p(\bar{x}, y) \to (\exists y \in \mathbf{S}^b_p)\mathcal{P}_p(\bar{x}, y)].$$

Proof: Fix $A \in \mathbf{S}_0$ such that $\bar{x} \in A$. Since **ZFC** holds in $\mathbf{S}_p$,

$$(\exists Z)(\forall \bar{z} \in A)[(\exists y)\mathcal{P}(\bar{z}, y) \to (\exists y \in Z)\mathcal{P}(\bar{z}, y)]$$

holds in $\mathbf{S}_p$. This is an $\in$-statement with the parameter $A \in \mathbf{S}_0$, so by Transfer it holds in $\mathbf{S}_0$. Fix $Z \in \mathbf{S}_0$ so that $(\forall \bar{z} \in A)[(\exists y)\mathcal{P}(\bar{z}, y) \to (\exists y \in Z)\mathcal{P}(\bar{z}, y)]$ holds in $\mathbf{S}_0$. By Transfer again, it holds in $\mathbf{S}_p$ as well. Since $\bar{x} \in \mathbf{S}_p \cap A$ and $(\exists y \in \mathbf{S}_p)\mathcal{P}_p(\bar{x}, y)$, we conclude that $(\exists y \in \mathbf{S}_p \cap Z)\mathcal{P}_p(\bar{x}, y)$. But $Z \in \mathbf{S}_0$, so such y is in $\mathbf{B}$ and we have $(\exists y \in \mathbf{S}^b_p)\mathcal{P}_p(\bar{x}, y)$. $\square$

Theorem 15 (p is a set or $p = \infty$)

$$(\forall \bar{x} \in \mathbf{S}^b_p)(\mathcal{P}_p(\bar{x}) \Leftrightarrow \mathcal{P}^b_p(\bar{x})).$$

Proof: We proceed by induction on the complexity of the statement. The only nontrivial step is when $\mathcal{P}$ is of the form $(\exists y)\mathcal{Q}(\bar{x}, y)$, $\bar{x} \in \mathbf{S}^b_p$ and $\mathcal{P}_p(\bar{x})$

holds, that is, $(\exists y \in \mathbf{S}_p)\mathcal{Q}_p(\bar{x}, y)$. Here we use Theoerm 14 to conclude that $(\exists y \in \mathbf{S}_p^b)\mathcal{Q}_p(\bar{x}, y)$. Fix such $y \in \mathbf{S}_p^b$. By the inductive assumption, $\mathcal{Q}_p^b(\bar{x}, y)$ holds, so $(\exists y \in \mathbf{S}_p^b)\mathcal{Q}_p^b(\bar{x}, y)$, i.e., $\mathcal{P}_p^b(\bar{x})$. □

We have all the ingredients for the proof of the final theorem.

Theorem 16
RBST *holds in the interpretation* $(\mathbf{B}, =, \in, \sqsubseteq)$.

Proof: *Relativization* is inherited from $\mathbf{RIST}^-$.

ZFC *in* $\mathbf{S}_p^b$ follows from **ZFC** in $\mathbf{S}_p$ and Theorem 15.

Boundedness follows immediately from the definition of $\mathbf{B}$ and the fact that $\mathbf{S}_0^b \subseteq \mathbf{S}_p^b$.

Transfer:
Let $\bar{x} \in \mathbf{S}_p^b$. Then $\mathcal{P}_p^b(\bar{x}) \Leftrightarrow \mathcal{P}_p(\bar{x})$ by Theorem 15, $\mathcal{P}_p(\bar{x}) \Leftrightarrow \mathcal{P}(\bar{x})$ by Transfer in $\mathbf{RIST}^-$, and $\mathcal{P}(\bar{x}) \Leftrightarrow \mathcal{P}^b(\bar{x})$, again by Theorem 15.

Inner Standardization:
Let $x \in \mathbf{B}$; fix $P \in \mathbf{S}_0^b$ such that $x \subseteq P$. By Inner Standardization in $\mathbf{RIST}^-$, there is $y \in \mathbf{S}_p$ such that $(\forall z \in \mathbf{S}_p)(z \in y \Leftrightarrow z \in x)$. Let $\widetilde{y} = y \cap P$. Then also $(\forall z \in \mathbf{S}_p)(z \in \widetilde{y} \Leftrightarrow z \in x)$, and $\widetilde{y} \in \mathbf{B}$.

Special Idealization:
This is inherited from $\mathbf{RIST}^-$: If $B \in \mathbf{B}$ and $y \in B$, then $y \in \mathbf{B}$.

We show in the Appendix (Theorem 17) that Inner Standardization and Special Idealization imply the full versions of Standardization and Bounded Idealization, respectively. This completes the proof of the theorem.

□

Corollary
RBST *is a conservative extension of* **ZFC**.

Appendix

We let $\mathbf{BST}^-$ denote an ostensibly weaker theory obtained from $\mathbf{BST}$ by replacing the axioms of Standardization and Bounded Idealization by, respectively,

Inner Standardization: $(\forall x)(\exists^{\mathbf{S}} y)(\forall^{\mathbf{S}} z)(z \in y \Leftrightarrow z \in x)$.

Special Idealization:

For all standard A, B and all $R \subseteq A \times B$,

$$(\forall^{\mathbf{S}} a \in \mathcal{P}^{\mathrm{fin}}(A))(\exists y \in B)(\forall x \in a)(\langle x, y\rangle \in R) \Leftrightarrow (\exists y \in B)(\forall^{\mathbf{S}} x \in A)(\langle x, y\rangle \in R).$$

Theorem 17
BST$^-$ *implies* ***BST***.

We begin by establishing the following very important result in $\mathbf{BST}^-$. The variable U always denotes an ultrafilter over $I = \bigcup U$. We write $x \,\mathbb{M} U$ (x is in the *monad* of U) for the statement

$$(\forall^{\mathbf{S}} X)(X \in U \rightarrow x \in X).$$

Theorem 18
(Reduction Theorem)
There is an effective procedure that assigns to each $\in$-$\mathbf{S}$-formula $\mathcal{P}(x_1, \ldots, x_k)$ an $\in$-formula $\mathcal{P}^s(U)$ such that, for all $x_1, \ldots, x_k$ and all standard U with $\langle x_1, \ldots, x_k\rangle \,\mathbb{M} U$ we have $\mathcal{P}(x_1, \ldots, x_k) \Leftrightarrow \mathcal{P}^s(U)$.
In particular,

$$\mathcal{P}(x_1, \ldots, x_k) \Leftrightarrow (\exists^{\mathbf{st}} U)(\langle x_1, \ldots, x_k\rangle \,\mathbb{M} U \ \wedge\ \mathcal{P}^s(U)) \Leftrightarrow$$
$$(\forall^{\mathbf{st}} U)(\langle x_1, \ldots, x_k\rangle \,\mathbb{M} U \rightarrow\ \mathcal{P}^s(U)).$$

The first result of this nature was proved by Nelson [9] for **IST** (Reduction Algorithm). Kanovei adapted it to **BST** (see [7]). The formulation given here is due to Andreev, who proved it in **BST** with a weak version of Standardization. The proof below is from [1].

Proof: Let $\mathcal{P}(x_1, \ldots, x_k)$ be an $\in$-$\mathbf{S}$-statement where all free variables are among $x_1, \ldots, x_k$. Renaming the bound variables if necessary, we can assume that all bound variables are distinct from all free variables and from each other (ie, if $Q_1 y_1$ and $Q_2 y_2$ are distinct occurences of quantifiers in $\mathcal{P}$, then y_1 and y_2 are distinct variables).

We proceed by induction on the complexity of $\mathcal{P}$. Let $1 \leq i, j \leq k$.
$(x_i \in x_j)^s$ is the statement "$\{\langle a_1, \ldots, a_k\rangle \in I = \bigcup U : a_i \in a_j\} \in U$";
$(x_i = x_j)^s$ is the statement "$\{\langle a_1, \ldots, a_k\rangle \in I = \bigcup U : a_i = a_j\} \in U$";
$(\mathbf{S}(x_i))^s$ is "$(\exists a)\ \{\langle a_1, \ldots, a_k\rangle \in I = \bigcup U : a_i = a\} \in U$";
$(\mathcal{P} \wedge \mathcal{Q})^s$ is $\mathcal{P}^s \wedge \mathcal{Q}^s$; $(\neg\mathcal{P})^s$ is $\neg\mathcal{P}^s$;
$((\exists y)\mathcal{Q}(x_1, \ldots, x_k, y))^s$ is $(\exists V)\ (\pi[V] = U \ \wedge\ \mathcal{Q}^s(V))$, where V is an ultrafilter over $I \times J = \bigcup V$ and $\pi : I \times J \rightarrow I$ is the projection mapping $\langle i, j\rangle \mapsto i$; $\pi[V] = U$ means that $(\forall X \subseteq I)(X \in U \Leftrightarrow X \times J \in V)$.

We verify the claim of the theorem in the last case. Let U be a standard ultrafilter over I and $x_1, \dots, x_k \, \mathbb{M} U$; note that $\langle x_1, \dots, x_k \rangle \in I$.

Assume that $(\exists y)\mathcal{Q}(x_1, \dots, x_k, y)$ and fix some y such that $\mathcal{Q}(x_1, \dots, x_k, y)$ holds. By Boundedness, $y \in J$ for some standard J. Let $W = \{Z \subseteq I \times J \mid \langle\langle x_1, \dots, x_k\rangle, y\rangle \in Z\}$. From Inner Standardization we get a standard V such that $Z \in V \Leftrightarrow Z \in W$, for all standard $Z \subseteq I \times J$. It is easy to check that V is an ultrafilter over $I \times J$, $\pi[V] = U$, and $\langle\langle x_1, \dots, x_k\rangle, y\rangle \, \mathbb{M} V$. Hence $\mathcal{Q}^s(V)$ holds by the inductive assumption.

For the converse assume that there exists V such that $\pi[V] = U \ \wedge \ \mathcal{Q}^s(V)$; by Transfer, we can take V to be standard. Let $\{Z_1, \dots, Z_n\}$ be a standard finite subset of V and $Z = \bigcap_{1 \le i \le n} Z_i$; note $Z \in V$ and Z is standard. As $\pi(Z) \in U$ (exercise), we have $\langle x_1, \dots, x_k \rangle \in \pi(Z)$, so there exists some $y \in J$ such that $\langle\langle x_1, \dots, x_k\rangle, y\rangle \in Z$. Using Special Idealization we obtain $y \in J$ such that $\langle\langle x_1, \dots, x_k\rangle, y\rangle \in Z$ holds for all standard $Z \in V$. [Let $A = V$, $B = J$ and $R(Z, y) \Leftrightarrow \langle\langle x_1, \dots, x_k\rangle, y\rangle \in Z$.] This just means that $\langle\langle x_1, \dots, x_k\rangle, y\rangle \, \mathbb{M} V$. By the inductive assumption, it now follows from $\mathcal{Q}^s(V)$ that $\mathcal{Q}(x_1, \dots, x_k, y)$ holds. Hence $(\exists y)\mathcal{Q}(x_1, \dots, x_k, y)$ holds. $\Box$

We can now prove Standardization and Bounded Idealization in $\mathbf{BST}^-$, and thus complete the proof of Theorem 17.

Theorem 19
BST$^-$ *proves Standardization.*

Proof: Let $\mathcal{Q}(z, x, \bar{x})$ be an $\in$-**S**- statement. By the Reduction Theorem, there is an $\in$-statement $\mathcal{Q}^s(U)$ such that $\mathcal{Q}(z, x, \bar{x}) \Leftrightarrow (\exists^{\mathbf{st}} U)(\langle z, x, \bar{x}\rangle \, \mathbb{M} U \ \wedge \ \mathcal{Q}^s(U))$. We fix a standard ultrafilter U_0 such that $\langle x, \bar{x}\rangle \, \mathbb{M} U_0$. Define the projection σ by $\langle z, x, \bar{x}\rangle \mapsto \langle x, \bar{x}\rangle$. It is easy to verify that, for any standard z and U, $\langle z, x, \bar{x}\rangle \, \mathbb{M} U \Leftrightarrow (U_0 \cap \sigma[U])$ is an ultrafilter $\wedge \{\langle w, v, \overline{v}\rangle \in \bigcup U : w = z\} \in U$. Using Transfer we have that, for standard z, $\mathcal{Q}(z, x, \bar{x}) \Leftrightarrow (\exists U)[(U_0 \cap \sigma[U])$ is an ultrafilter $\wedge \{\langle w, v, \overline{v}\rangle \in \bigcup U : w = z\} \in U \ \wedge \ \mathcal{Q}^s(U)]$, and the statement on the right side is an $\in$-statement (with a standard parameter U_0).

Let $x \in A$ where A is standard. Given $\mathcal{P}(z, x, \bar{x})$, let $\mathcal{Q}(z, x, \bar{x})$ be the statement $z \in x \wedge \mathcal{P}(z, x, \bar{x})$. By the Axiom of Separation of **ZFC**, the set of all $z \in A$ that satisfy the equivalent $\in$-statement exists and is standard. It has the property required by Standardization. $\Box$

Theorem 20
BST$^-$ *proves Bounded Idealization.*

Proof: Assume that the left side holds. Axiom of Choice implies the existence of a set B such that, for every $a \in \mathcal{P}^{\text{fin}}(A)$, if $(\exists y)(\forall x \in a)\mathcal{P}(x, y, A, \bar{x})$, then $(\exists y \in B)(\forall x \in a)\ \mathcal{P}(x, y, A, \bar{x})$; by Boundedness, we can take B to be standard. Define $R := \{\langle x, y\rangle \in A \times B : \mathcal{P}(x, y, A, \bar{x})\}$ and apply Special Idealization to obtain the right side.

The converse implication is trivial, because if $a \subseteq A$ is standard and finite, then all $x \in a$ are standard elements of A. $\Box$

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