# Construction of Number Systems Including Ultralarge and Ultrasmall Numbers

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November 8, 2014

In this note we go through the construction of number systems and show the main properties used in analysis with ultrasmall numbers (AUN) from first principles. These properties are the existence of ultralarge integers, ultralarge and ultrasmall rational numbers and real numbers and the existence of the observable neighbour in the real numbers. The classical part of the construction will be described but not given in detail and can be found in [1],[4] or [3]. It goes as follows: from the axioms, the existence of $\mathbb{N}$ as the least inductive set is shown. Using the specific axioms of AUN, it is proven that there exist ultralarge natural numbers. Then equivalence classes of ordered pairs of natural numbers yield integers (among which some are ultralarge). Equivalence classes of ordered pairs of integers in turn yield rational numbers. The existence of ultralarge and ultrasmall rational numbers follows immediately. The real numbers are constructed as equivalence classes of Cauchy sequences of rational numbers and their completeness is proven. This enables us to show the existence of an observable neighbour for any number which is not ultralarge.

The extra axioms of AUN, when stated in full generality, are technically rather involved. They are stated here in a form fit for dealing with sets of ordered numbers, in particular they are given with respect to one variable only. We assume some familiarity with working with ultrasmall numbers.

We summarise the extra principles first.

## 1 Principles of Analysis with Ultrasmall Numbers

**Definition 1.** *The* ***context*** *of a property, function or set, is the list of parameters used in its definition. The context can be a single parameter or even empty.*

**Observability Principle**

- A set is observable relative to a context if it is observable relative to at least one parameter of the context.
- Every set is observable relative to some context.
- $\emptyset$ is observable relative to every context.
- Two sets $a$ and $b$ will always have a common context. If $a$ is not observable relative to $b$, then $b$ will be observable relative to $a$.

If a set is observable relative to every context, we say that it is always observable or that it is standard.

**Definition 2.**

1. *A statement is internal if it does not refer to observability or the context of every relative concept that occurs in it is given by the parameters of the statement.*
2. *An internal concept is a concept defined by an internal statement.*
3. *Previously defined internal concepts can be used in subsequent internal statements.*

**Definition Principle**
Internal defining statements can be used to define sets and functions.

**Idealisation**
Given a set $A$ and a property $P$ not referring to observability.
For every observable finite subset of A there is a $y$ such that the property $P(x, y, A)$ holds for every $x$ in the finite subset

if and only if

there is a $y$ such that $P(x, y, A)$ holds for every observable $x$ in $A$.

If $x$ is observable relative to $a$, we say that $x$ is $a$-observable.

**Standardisation** Consider two contexts $a$ and $b$, where $b$ is less observable than $a$.
Given a $b$-observable set $B$, then there is an $a$-observable set $A$ such that all $a$-observable elements of $B$ are exactly the $a$-observable members of $A$.

Let P be an internal statement and $c$ its context. Then $P^c$ is the statement obtained from P by replacing every $\exists$ by $\exists^c$ (meaning "there is an observable") and every $\forall$ by $\forall^c$ (meaning "for all observable"), i.e. where every quantifier is restricted to observability. Then:

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| **Transfer** For all observable x, the internal property P(x) holds if and only it holds when all of its quantifiers are restricted to observability. |

## 2 Natural numbers

The existence of a set $E$ such that $\emptyset \in E$ and $x \in E \Rightarrow x \cup \{x\} \in E$ (an inductive set) is given by the infinity axiom of ZFC.

The set $\mathbb{N}$ is the least inductive set (see [1]) and its existence follows from the axioms of ZFC.[(1)] Addition is defined in the usual way. Hence 0 is identified with $\emptyset$, then $1 = \{\emptyset\}$, $2 = 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\}$ and $n+1 = n \cup \{n\}$.

The observability principle states that $\emptyset$ is standard. This is equivalent to stating that 0 is standard.

**Theorem 1** (Closure)**.** *Given an internal statement $P$, if there is an $x$ such that $P(x)$, then there is an observable $x$ such that $P(x)$.*

*Proof.* Consider a statement $P(x)$ – its context contains the parameters of $P$ and $x$ – and the statement $\exists x P(x)$. By transfer $\exists^c x P^c(x)$ also holds. Fix $x$ such that $P^c(x)$ holds. By transfer once more, $P(x)$ holds as well, hence $\exists^c x P(x)$. □

If a set is defined with no parameters, then it is always observable, or standard. The axioms of set theory have no parameters and determine therefore standard sets. Hence:

**Theorem 2.** $\mathbb{N}$ *is a standard set.*

**Theorem 3.** *The set $\{x \in \mathbb{N} \mid x \leq k\}$ is as observable as $k$.*

*Proof.* The parameter used in defining this set is $k$ which determines the context hence, by closure, the observability of the set. □

**Theorem 4.** *There exist nonstandard natural numbers.*

(1) axioms of Zermelo and Fraenkel with the addition of the axiom of choice: this is the classical setting for set theory

*Proof.* Since 0 is standard and by closure, 1 and 2 are standard, we know that there are standard sets of numbers. The set $A$ used in the idealisation principle will be $\mathbb{N}$. For every finite standard subset $a \subset \mathbb{N}$, there is a $y$ such that for every $x \in a$, the property "$y \in \mathbb{N}$ and $y > x$" holds (take $y = \max\{a\} + 1$) Hence, by idealisation, there is a $y$ such that "$y \in \mathbb{N}$ and $y > x$" holds for all observable $x \in \mathbb{N}$. Obviously, this $y$ cannot be itself standard. □

**Theorem 5.** *Relative to any context, there exist numbers which are not observable.*

This is simply a restatement of theorem 4 where standardness is replaced by observability relative to any given context.

*Proof.* By theorem 3, we define $\{n \in \mathbb{N} \mid n \leq y\}$ using the nonstandard $y$ determined above . The same proof as for theorem 4, referring to observability of $y$ instead of standardness, yields a numbers which is not observable in this extended context. □

**Definition 3.** *Relative to a context; if a natural number is greater than any observable natural number, then it is ultralarge.*

By theorem 4, relative to some context, ultralarge numbers exist and are non observable. We now show that non observable numbers are ultralarge.

**Theorem 6.** *Relative to a context, if $k \in \mathbb{N}$ is observable and $n < k$ (with $n \in \mathbb{N}$), then $n$ is observable.*

*Proof.* We use idealisation in the contrapositive form:

*There is an observable finite subset of $A$ such that for every $y$ there is an $x$ in the finite subset such that $P(x, y, A)$ holds,*

*if and only of*

*for all $y$, there is an observable $x$ in $A$ such that the property holds.*

Since $A = \{n \in \mathbb{N} \mid n \leq k\}$ is a finite set, any subset of $A$ is finite. Let $P$ be the property "$y \in A \Rightarrow y = x$." The observable finite subset of $A$ can be $A$ itself, so the first part of idealisation holds, hence the property holds for every observable $x$ in $A$, which means that for every $y \in A$ there is an observable $x$ in $A$ such that $y = x$ i.e., all elements of $A$ are observable. □

By theorems 5 and 6, we obtain a general relation for natural numbers, that given a context, a number is non observable if and only if it is ultralarge. Idealisation has been used twice to produce this result and will be used no more in this setting.

### Internal or standard view?

A question sometimes asked about these ultralarge natural numbers (and later, about the ultrasmall real numbers) is whether they are the *same* natural numbers as the ones used by those who do not have this concept of observability. We first stress that this question is not mathematical but philosophical.

The internal view is that they are the same. As seen above, the set of natural numbers follows from the axioms of ZFC with no extra considerations. This justifies the claim that they are the same. Then idealisation is used to show that within the set just constructed, ultralarge numbers are *already* there. The understanding is that idealisation is an extension to the syntax which enables us to say more things about sets. We do not construct new objects but we use extra properties to describe them. No new object is constructed because the only properties allowed to define a set are the internal ones: those that do not refer to observability or those that refer contextually to observability – those, in fact, which use observability for dummy variables only.

The standard view is that idealisation *produces* new objects – the ultralarge natural numbers. The existence quantifier is understood as "*now* there exist also." But since the complete set of natural numbers – the usual ones and the ultralarge ones – is described without referring to observability, by closure, the name "natural numbers" must be used to describe the whole: new and old numbers together.

Whichever view is adopted, the mathematics are the same.

## 3 Integers

The set $\mathbb{Z}$ is the set of equivalence classes of ordered pairs of natural numbers with the equivalence relation $(a, b) \sim (c, d)$ if $a+d = b+c$ (see [4]). Addition is defined by $(a, b) + (c, d) = (a + c, b + d)$ and it is left as an exercise to show that every pair is equivalent to a pair either of the form $(n, 0)$ or of the form $(0, n)$, the canonical representatives. Multiplication is given by $(a, b) \cdot (c, d) = (ac + bd, ad + bc)$. For more details, see [4].

We identify a natural number $n$ with a the pair $(n, 0)$. Negative integers are equivalent to pairs of the form $(0, n)$.

These definitions do not refer to observability, hence:

**Theorem 7.** *$\mathbb{Z}$ is a standard set.*

**Definition 4** (Absolute value)**.** *Given canonical representatives* $(0, n)$ *or* $(n, 0)$*, their absolute value is* $n$

We now redefine:

**Definition 5** (Ultralarge numbers)**.** *Relative to a context; if a number is greater in absolute value than any observable number, then it is ultralarge.*

It is immediate that, relative to a context, there are positive ultralarge integers and negative ultralarge integers.

## 4 Rational numbers

The set $\mathbb{Q}$ is the set of equivalence classes of pairs of integers with the equivalence relation $(a, b) \sim (c, d)$ if $a \cdot d = b \cdot c$ (for $b \neq 0 \neq d$) with addition and multiplication performed the usual way with fractions.

These definitions do not refer to observability, hence:

**Theorem 8.** $\mathbb{Q}$ *is a standard set.*

The ordering of rational numbers allows to define that a rational number is ultralarge if it is greater in absolute value than any observable integer and that there are ultralarge positive rationals and ultralarge negative rationals.

**Definition 6** (Ultrasmall numbers)**.** *Relative to a context; if a number is smaller in absolute value than any non zero observable number, then it is ultrasmall.*

**Lemma 1.** *Relative to some context. If* $M$ *is ultralarge and* $n \neq 0$ *is observable, then* $n \cdot M$ *is ultralarge.*

*Proof.* wlog assume $n, M$ positive. Then $n \cdot M > M$ so $n \cdot M$ is ultralarge. □

**Lemma 2.** *If* $x$ *rational is observable then* $x = p/q$ *for observable* $p, q$*.*

(Immediate by taking $p$ and $q > 0$ mutually prime, because such are uniquely determined by $x$, and hence observable).

**Theorem 9.** *Relative to some context; if* $M \in \mathbb{N}$ *is ultralarge, then* $\frac{1}{M}$ *is an ultrasmall rational number.*

*Proof.* wlog, assume $M > 0$ and assume, for a contradiction, that $\frac{1}{M}$ is not ultrasmall i.e., there is a rational observable non zero $\frac{p}{q}$ such that $0 < \frac{p}{q} < \frac{1}{M}$, with $p, q$ observable by lemma 2. But then $0 < M \cdot p < q$ and by lemma 1, $p \cdot M$ is ultralarge and cannot be less than $q$. □

If $a - b$ is ultrasmall or zero, we write $a \simeq b$ and say that $a$ is ultraclose to $b$.

The other rules of "ultracalculus" follow immediately (that if $\varepsilon$ is ultrasmall and $a$ is observable, then $\varepsilon \cdot a \simeq 0$, etc.) and can be found in [2] p12.

# 5 Real numbers

**Definition 7.** *Let $(u_n)_{n \geq k}$ be a sequence of rational numbers. We say that $(u_n)_{n \geq k}$ is a **Cauchy sequence** if*

$$u_{N'} \simeq u_N, \quad \textit{for all positive ultralarge integers } N, N'.$$

A context is given by the sequence. With this definition, the classical construction of real numbers as equivalence classes of Cauchy sequences is performed. Two Cauchy sequences $(u_n)$ and $(v_n)$ are equivalent if $u_N \simeq v_N$ for all ultralarge $N$ (the context is given by the two sequences). The definition principle ensures that the equivalence classes are sets. The existence of ultrasmall and ultralarge real numbers follow by extending the definitions naturally to include real numbers.

The last important property of real numbers which is needed for analysis is the existence of the observable neighbour. We must first show that the set of real numbers is complete in the sense that if a set of real numbers is bounded above then it has a least upper bound. We show it in a restricted form for sets which are intervals with an upper bound but no lower bound i.e., if $x \in A$ and $y < x$ then $y \in A$.

**Theorem 10.** *Let $A \subset \mathbb{R}$ be such if $x \in A$ and $y < x$ then $y \in A$, and there exists $b \notin A$. Then $A$ has a least upper bound.*

*Proof.* By definition of $A$, $x \in A \Rightarrow x < b$ so $b$ is an upper bound. By closure, since there is an upper bound, there is an observable upper bound. Hence wlog, we may assume that $b$ is observable. And since there is an $x \in A$ we consider an observable $a \in A$. (Observability is determined by $A$.)

Then for $n \in \mathbb{N}$ we define $a_k = a + k \cdot \frac{b-a}{n}$ for $k = 0, 1, ..., n$ and $c_n = \max\{a_k | a_k \in A\}$ and $d_n = \min\{a_k | a_k \notin A\}$. We immediately have $d_n - c_n = \frac{b-a}{n}$.

$\{d_n\}_n$ is a decreasing sequence of upper bounds and $\{c_n\}_n$ is an increasing sequence of elements of $A$. The sequences are equivalent since for ultralarge $N \in \mathbb{N}$ we have $d_N - c_N = \frac{b-a}{N} \simeq 0$. These sequences clearly define a least upper bound.[2] $\square$

**Theorem 11** (Observable neighbour)**.** *If $x$ is not ultralarge, then there is a unique observable $a$ and $h \simeq 0$ such $x = a + h$.*

We use the standardisation but first give an example of its consequences. Consider the $a = 1$ and $b = n$ ultralarge and $[0, n]$. There is a standard set containing all standard elements of $[0, n]$ and that set is the set of all non-negative reals. This is the standardisation of $[0, n]$.

**Theorem 12.** *The standardisation of a set is unique.*

*Proof.* Assume there are two standardisations $A$ and $B$ of a set $C$. $A$ and $B$ have same observability and contain the same observable elements, hence by closure all their elements are the same. $\square$

*Proof.* [of theorem 11]

Fix a context and a real number $x$ not ultralarge. Wlog assume $x > 0$. Consider the set $\{u \in \mathbb{R} \mid u \leq x\}$. This set has a unique standardisation $A$ (not empty since it contains 0) and is bounded above by an observable number since $x$ is not ultralarge. Therefore $A$ has a least upper bound $a$ – which is observable (by closure).

We now show that $x \simeq a$, so $a$ is the observable neighbour of $x$.

If not, then $|a-x| > s > 0$ for some observable $s$. This means that either $x > a+s$ or $x < a-s$. In the first case, $a+s \in A$, contradicting $a = \sup A$. In the second case, $a - s$ is an upper bound on $A$, again contradicting $a = \sup A$. $\square$

One can also prove uniqueness of the observable neighbour by assuming that there are two observable numbers, $a$ and $b$, satisfying $a \simeq x \simeq b$. This implies $a \simeq b$ which in turn implies $a-b \simeq 0$. By closure, $a-b$ is observable, hence not ultrasmall, so $a - b = 0$ which is always observable.

[2] There are other constructions for the real numbers, with Dedekind cuts for instance. The resulting fields are, as it is well known, the same up to isomorphism, which means that essentially there is only one set of real numbers (see [1]) – which again shows why the internal view is reasonable.

## References

[1] Karel Hrbacek and Thomas Jech. *Introduction to Set Theory. Third Edition, Revised and Expanded.* Marcel Dekker, Inc., 1999.

[2] Karel Hrbacek, Olivier Lessmann and Richard O'Donovan *Analysis with ultrasmall numbers* Chapman and Hall, CRC Press, 2014.

[3] Michael Spivak *Calculus. Third edition.* Publish or Perish Inc. 1994.

[4] Howard Levi. *Elements of Algebra, fourth edition.* Chelsea Publishing, 1961.