Stability implies Closure.

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In the Appendix to [1], both Stability and Closure are derived from the axioms of **RIST**. Here we show that Closure is a direct consequence of Stability. We follow the terminology and notation of the Appendix.

Internal statements are statements of the form

$$\mathcal{P}(x_1, \ldots, x_k; \mathbf{S}_{\langle x_1,\ldots,x_k\rangle}).$$

Stability can be formulated as follows:

$$\mathcal{P}(x_1, \ldots, x_k; \mathbf{S}_{\langle x_1,\ldots,x_k\rangle}) \Leftrightarrow \mathcal{P}(x_1, \ldots, x_k; \mathbf{S}_{\langle x_1,\ldots,x_k,y_1,\ldots,y_\ell\rangle}).$$

Closure Principle asserts:

$$(\exists x)\mathcal{P}(x, x_1, \ldots, x_k; \mathbf{S}_{\langle x,x_1,\ldots,x_k\rangle}) \Rightarrow (\exists x \in \mathbf{S}_{\langle x_1,\ldots,x_k\rangle})\mathcal{P}(x, x_1, \ldots, x_k; \mathbf{S}_{\langle x,x_1,\ldots,x_k\rangle}).$$

Stability implies Closure

Let us assume that $(\exists x)\mathcal{P}(x, x_1, \ldots, x_k; \mathbf{S}_{\langle x,x_1,\ldots,x_k\rangle})$. Fix some x for which $\mathcal{P}(x, x_1, \ldots, x_k; \mathbf{S}_{\langle x,x_1,\ldots,x_k\rangle})$ holds and fix some y such that $x \in \mathbf{S}_{\langle x_1,\ldots,x_k,y\rangle}$ (for example, $y = x$). By Stability, $\mathcal{P}(x, x_1, \ldots, x_k; \mathbf{S}_{\langle x,x_1,\ldots,x_k,y\rangle})$ holds. As $\mathbf{S}_{\langle x,x_1,\ldots,x_k,y\rangle} = \mathbf{S}_{\langle x_1,\ldots,x_k,y\rangle}$, we have

$$(\exists x \in \mathbf{S}_{\langle x_1,\ldots,x_k,y\rangle})\mathcal{P}(x, x_1, \ldots, x_k; \mathbf{S}_{\langle x_1,\ldots,x_k,y\rangle}).$$

By Stability again,

$$(\exists x \in \mathbf{S}_{\langle x_1,\ldots,x_k\rangle})\mathcal{P}(x, x_1, \ldots, x_k; \mathbf{S}_{\langle x_1,\ldots,x_k\rangle}).$$

But $\mathbf{S}_{\langle x,x_1,\ldots,x_k\rangle} = \mathbf{S}_{\langle x_1,\ldots,x_k\rangle}$ for $x \in \mathbf{S}_{\langle x_1,\ldots,x_k\rangle}$, and we get also

$$(\exists x \in \mathbf{S}_{\langle x_1,\ldots,x_k\rangle})\mathcal{P}(x, x_1, \ldots, x_k; \mathbf{S}_{\langle x,x_1,\ldots,x_k\rangle}).$$

□

References

[1] K Hrbacek, O Lessmann and R O'Donovan, *Analysis with ultrasmall numbers*, Chapman-Hall/CRC Press, 2014.